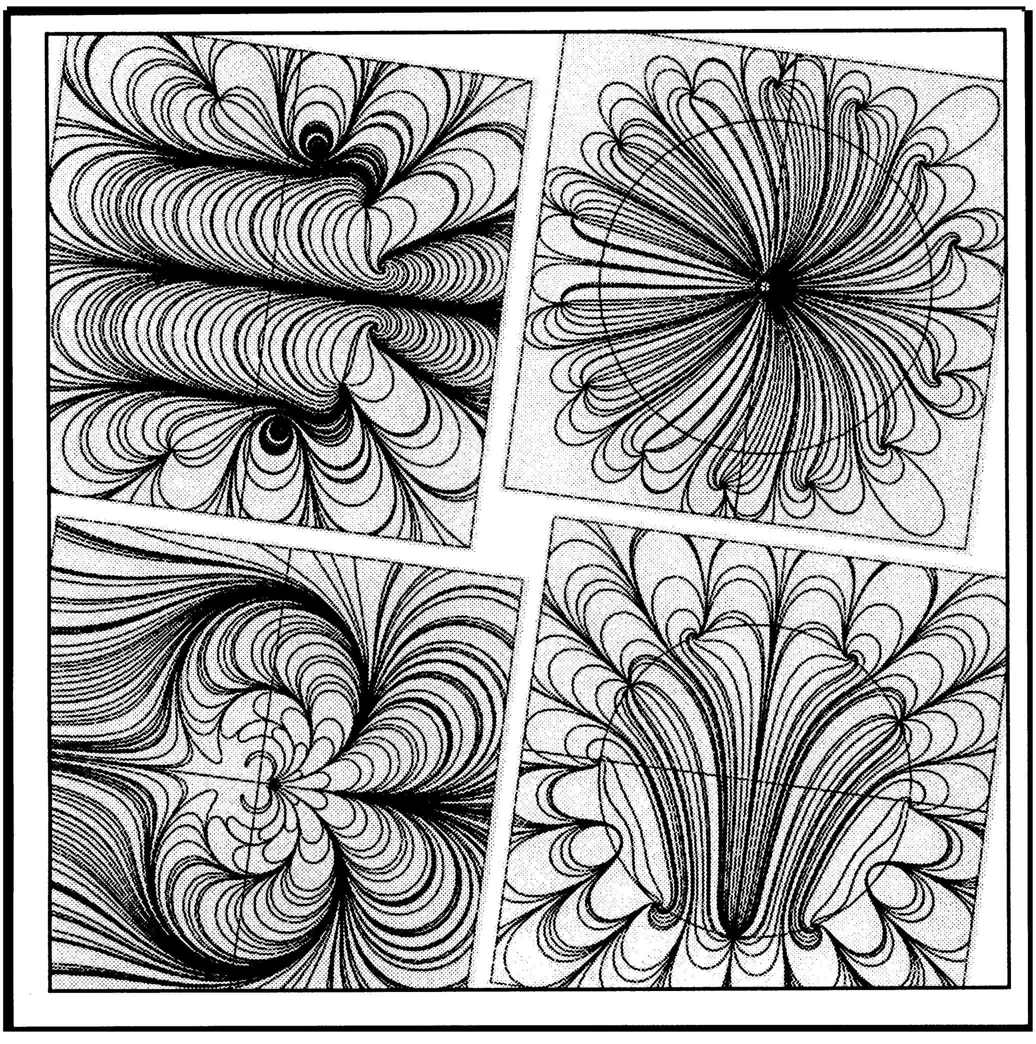
Vol. 69, No. 1 February 1996

MATHEMATICS MAGAZINE



- Beyond Sin and Cos
- A Round-Up of Square Problems
- On Using Flows to Visualize Functions of a Complex Variable

An Official Publication of The MATHEMATICAL ASSOCIATION OF AMERICA

EDITORIAL POLICY

Mathematics Magazine aims to provide lively and appealing mathematical exposition. This is not a research journal and, in general, the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for an article for the *Magazine*. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Articles on pedagogy alone, unaccompanied by interesting mathematics, are not suitable. Manuscripts on history are especially welcome, as are those showing relationships between various branches of mathematics and between mathematics and other disciplines.

A more detailed statement of author guidelines appears in this *Magazine*, Vol. 69, pp. 78–79, and is available from the Editor. Manuscripts to be submitted should not be concurrently submitted to, accepted for publication by, or published by another journal or publisher.

Send new manuscripts to Paul Zorn, Editor, Department of Mathematics, St. Olaf College, 1520 St. Olaf Avenue, Northfield, MN 55057-1098. Manuscripts should be laser printed, with wide spacing, and prepared in a style consistent with the format of *Mathematics Magazine*. Authors should submit three copies and keep one copy. In addition, authors should supply the full five-symbol Mathematics Subject Classification number, as described in *Mathematical Reviews*, 1980 and later. Copies of illustrations should be supplied on separate sheets, both with and without lettering added.

Cover illustration: A collage of complex flows (see pp. 28–34). By Kristine L. Remer, a junior art student at St. Olaf College, Northfield, Minn.

AUTHORS

Duane DeTemple received his B.S. in Applied Science and Mathematics from Portland State College in 1964. He then began graduate studies in mathematics at Stanford University and, following a year off working as an engineer at Lockheed Missiles & Space Company, completed his doctoral dissertation in univalent function theory in 1970. He has taught at Washington State University since then, pursuing interests in analysis, graph theory, combinatorics, and mathematics education. He is co-author with Calvin Long of the new text *Mathematical Reasoning for Elementary Teachers*, Harper-Collins, 1996. His nonacademic interests include tennis, home brewing, hiking, canoeing, and playing recorder and capped reed instruments in an early music consort.

Sonia Harold received her B.S. in mathematics from Washington State University in December of 1995. She is especially interested in the field of optimization, which she hopes to put to use working in industry. She also has a teaching certificate, and at a later time intends to teach high school mathematics. Her hobbies include playing flute and piano and speed walking.

Martin E. Muldoon (Ph.D., Alberta, 1966) is Professor and Director of the Applied Mathematics section in the Department of Mathematics and Statistics at York University in Metropolitan Toronto.

Abraham A. Ungar (Ph.D., Tel-Aviv University, 1973) is Professor of Mathematics at North Dakota State University in Fargo. Since 1978, when Muldoon invited Ungar (then at Rhodes University in South Africa) to give a talk at York on generalized trigonometric functions, both authors have been aware that these elementary functions "beyond sin and cos" have repeatedly been rediscovered and that they deserve to be entered into the mainstream literature to be part of the lore learned by all mathematics students.

Vol. 69, No. 1 February 1996

MATHEMATICS MAGAZINE

The *MATHEMATICS MAGAZINE* (ISSN 0025-570X) is published by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, D.C. 20036 and Montpelier, VT, bimonthly except July/August.
The annual subscription price for the MATHEMATICS MAGAZINE to an individual member of the Association is $16 included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $64. Student and unemployed members receive a 66% dues discount; emeritus members receive a 50% discount; and new members receive a 40% dues discount for the first two years of membership.) The nonmember/library subscription price is $68 per year.
Subscription correspondence and notice of change of address should be sent to the Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036. Microfilmed issues may be obtained from University Microfilms International, Serials Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

Advertising correspondence should be addressed to Ms. Elaine Pedreira, Advertising Manager, The Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036.

Second class postage paid at Washington, D.C. and additional mailing offices.

Postmaster: Send address changes to Mathematics Magazine Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036-1385.

PRINTED IN THE UNITED STATES OF AMERICA

ARTICLES

Beyond Sin and Cos

MARTIN E. MULDOON[1]
York University
North York, Ontario M3J 1P3, Canada

ABRAHAM A. UNGAR
North Dakota State University
Fargo, ND 58105

1. Introduction

The higher-order circular and hyperbolic functions deserve to be better known. Here we give their main properties in order to make them more accessible to teachers and students in calculus, linear algebra and differential equations courses. The study of these functions can be related to such diverse topics as the binomial theorem and the fast Fourier transform.

Here, for each positive integer n, we define r functions $F_{n,r}^{\alpha}(x)$, $r = 0, 1, \ldots, n-1$. The cases $\alpha = 1$ and $\alpha = -1$ correspond, respectively, to what are usually known as generalized hyperbolic functions and generalized circular or trigonometric functions. We find it useful to retain the parameter α; the case $\alpha = 0$ also gives something interesting.

The functions considered here are *elementary* and can be a rich source for student projects and investigations.

2. Background

The trigonometric functions can be generalized in many ways, some of them indispensable in the applications of mathematics. We mention, for example, the Bessel functions **[40]**, special cases of which have been around since the time of Euler, the hypergeometric functions **[30]** and their various generalizations. But the deeper study of these functions becomes difficult very quickly and so they are little studied except by those who need them for some application. It is of interest therefore to note that there exists a class of functions that preserve the elegance and simplicity of the trigonometric functions and that is easily presented to students in elementary courses. These generalized circular and hyperbolic functions have been rediscovered often since the first recorded account by Vincenzo Riccati in 1757. They preserve the flavor of striking results like Euler's formula

$$e^{ix} = \cos x + i \sin x, \tag{1}$$

[1]The work of this author was supported by a grant from the Natural Sciences and Engineering Research Council (Canada).

the determinantal identity

$$\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1, \tag{2}$$

and the matrix identity

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix} = \begin{pmatrix} \cos(x+y) & \sin(x+y) \\ -\sin(x+y) & \cos(x+y) \end{pmatrix}. \tag{3}$$

They also satisfy differential equations similar to the equations $y' - y = 0$, $y'' - y = 0$, and $y'' + y = 0$, satisfied by the exponential, hyperbolic, and trigonometric functions, respectively.

In spite of their simplicity and the length of time they have been around, the generalized circular functions are seldom discussed in textbooks. (An exception is **[14]**, pp. 336–339). This may be because they do not seem to have obvious applications. (See, however, the recent references **[28]**, **[42]**, **[20]**, **[15]**, **[9]**.) We believe that the value of the study of these functions lies rather in their compelling intrinsic beauty and in providing a rich source of examples and motivation in various elementary courses.

3. The Generalized Hyperbolic Functions

We can define the exponential function by the usual sum

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

In the case of the hyperbolic functions we take every second term in this sum:

$$\sinh(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \dots,$$

$$\cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \dots .$$

An obvious way to generalize this is to take every nth term in the sum. Thus, for any positive integer n, we define the n α-hyperbolic functions of order n, $F^{\alpha}_{n,r}$, on the real line (or in the complex plane) by the infinite power series,

$$F^{\alpha}_{n,r}(x) = \sum_{k=0}^{\infty} \frac{\alpha^k}{(nk+r)!} x^{nk+r}$$

$$= \frac{x^r}{r!} + \frac{\alpha x^{n+r}}{(n+r)!} + \frac{\alpha^2 x^{2n+r}}{(2n+r)!} + \dots,$$

$$r = 0, 1, \dots, n-1,$$

where α is real (or complex). For consistency we take

$$F^{\alpha}_{n,0}(0) = 1.$$

The function $F^{\alpha}_{n,r}$ is called the α-hyperbolic function of order n and rth kind.

There is a single α-hyperbolic function of order 1; it is the exponential function $F^{\alpha}_{1,0}(x) = e^{\alpha x}$. There are two α-hyperbolic functions of order 2; these are the functions $F_{2,0}^{\alpha}(x)=\cosh(\sqrt{\alpha}\,x)$ and $F_{2,1}^{\alpha}(x)=(1/\sqrt{\alpha})\sinh(\sqrt{\alpha}\,x)$, where $\sqrt{\alpha}$ is an arbitrarily specified square root of α, giving rise to the circular functions ($\alpha=-1$) and the hyperbolic functions ($\alpha=1$) as special cases. Similarly, there are three α-hyperbolic functions of order three, and so on.

Writing the infinite series in the definition of $F_{n,r}^{\alpha}$ explicitly, one may readily find the relationship

$$F_{n,r}^{\alpha}(x)=\left(\sqrt[n]{\alpha}\right)^{-r}\left(\sqrt[n]{\alpha}\,x\right), \qquad (\alpha\neq 0) \tag{4}$$

where $\sqrt[n]{\alpha}$ is an arbitrarily specified nth root of α. Although (4) shows that the case $\alpha\neq 0$ can be reduced to the case $\alpha=1$, retaining the α preserves a certain elegance in the formulas. It is also interesting to see, in Section 5, how the case $\alpha=0$ gives rise to polynomials; without the explicit use of the parameter α, this connection might go undetected. Like the circular and hyperbolic functions, the α-hyperbolic functions are generated by particularly simple differential equations normalized by "natural" initial conditions. The α-hyperbolic function of order n and rth kind $F_{n,r}^{\alpha}(x)$ satisfies the differential equation

$$f^{(n)}(x)=\alpha f(x) \tag{5}$$

normalized by the initial conditions

$$f^{(k)}(0)=\begin{cases}0, & k\neq r,\ 0\leq k\leq n-1,\\ 1, & k=r.\end{cases}$$

Moreover, differentiation permutes the α-hyperbolic functions cyclically, apart from a factor α in one case:

$$\frac{d}{dx}F_{n,r}^{\alpha}(x)=\begin{cases}F_{n,r-1}^{\alpha}(x), & 0<r\leq n-1,\\ \alpha F_{n,n-1}^{\alpha}(x), & r=0.\end{cases}$$

4. The Generalized Euler Formula

Our definition can be shown to lead to the generalized Euler formula

$$e^{\sqrt[n]{\alpha}\,x}=\sum_{r=0}^{n-1}\left(\sqrt[n]{\alpha}\right)^{r}F_{n,r}^{\alpha}(x) \tag{6}$$

where $\sqrt[n]{\alpha}$ is an arbitrarily specified nth root of α. Obviously, this reduces to (1) in case $n=2$, $\alpha=-1$. Since there are n nth roots of α, we see that (6) is actually a system of n linear equations. In Section 9, we will use the Fourier matrix to show that the system (6) can be solved for the $F_{n,r}^{\alpha}$, $r=0,\ldots,n-1$, to give

$$F_{n,r}^{\alpha}(x)=\frac{1}{n}\left(\sqrt[n]{\alpha}\right)^{-r}\sum_{k=0}^{n-1}\omega_n^{-rk}\exp\left[\omega_n^{k}\sqrt[n]{\alpha}\,x\right]. \tag{7}$$

Here $\omega_n=\exp[2\pi i/n]$ is a primitive nth root of unity. Actually, in the cases $\alpha=\pm 1$, (7) has been used to *define* the α-hyperbolic functions; see, e.g., **[13]**, p. 212.

Formula (7) may be used to express all the α-hyperbolic functions in terms of trigonometric and exponential functions. In the case $\alpha=1$, the three α-hyperbolic functions of order 3 are

$$F_{3,0}^{1}(x) = \frac{1}{3}\left[e^{x} + 2e^{-x/2}\cos\frac{\sqrt{3}\,x}{2}\right],$$

$$F_{3,1}^{1}(x) = \frac{1}{3}\left[e^{x} - 2e^{-x/2}\cos\left(\frac{\sqrt{3}\,x}{2} + \frac{\pi}{3}\right)\right],$$

and

$$F_{3,2}^{1}(x) = \frac{1}{3}\left[e^{x} - 2e^{-x/2}\cos\left(\frac{\sqrt{3}\,x}{2} - \frac{\pi}{3}\right)\right].$$

The first of these follows easily from (7). The others can be obtained then from (7) or by successive differentiation of the first one. FIGURE 1, produced using GNUPLOT[2], gives graphs of the hyperbolic functions $F_{3,k}^{1}(x)$ for $k = 0, 1, 2$. In the case $n = 4$, we

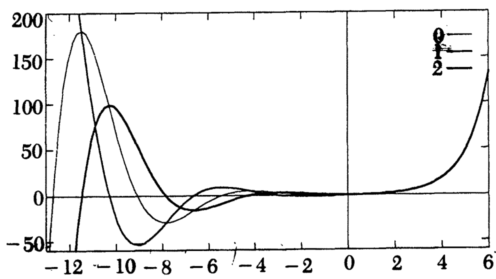


FIGURE 1
$F_{3,k}^{1}(x)$ vs. $x, k = 0, 1, 2$.

get the elegant formulas

$$\begin{aligned} F_{4,0}^{1}(x) &= (1/2)(\cosh x + \cos x), \\ F_{4,1}^{4}(x) &= (1/2)(\sinh x + \sin x), \\ F_{4,2}^{1}(x) &= (1/2)(\cosh x - \cos x), \\ F_{4,3}^{1}(x) &= (1/2)(\sinh x - \sin x). \end{aligned} \tag{8}$$

5. The α-hyperbolic Matrix

To exhibit the generalization of identities (1) and (2) provided by the α-hyperbolic functions we define the $n \times n$ α-hyperbolic matrix $H_n^{\alpha}(x)$ by the equation

$$H_n^{\alpha}(x) = \begin{bmatrix} F_{n,0}^{\alpha}(x) & F_{n,1}^{\alpha}(x) & F_{n,2}^{\alpha}(x) & \dots & F_{n,n-1}^{\alpha}(x) \\ \alpha F_{n,n-1}^{\alpha}(x) & F_{n,0}^{\alpha}(x) & F_{n,1}^{\alpha}(x) & \dots & F_{n,n-2}^{\alpha}(x) \\ \alpha F_{n,n-2}^{\alpha}(x) & \alpha F_{n,n-1}^{\alpha}(x) & F_{n,0}^{\alpha}(x) & \dots & F_{n,n-3}^{\alpha}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha F_{n,1}^{\alpha}(x) & \alpha F_{n,2}^{\alpha}(x) & \alpha F_{n,3}^{\alpha}(x) & \dots & F_{n,0}^{\alpha}(x) \end{bmatrix}. \tag{9}$$

The α-hyperbolic matrix $H_n^{\alpha}(x)$ is *factor-circulant*, that is, it is a matrix obtained by multiplying by α each of the elements below the main diagonal in a circulant matrix. Factor-circulants are considered by Ruiz Claeyssen and dos Santos Leal in **[28]**; see

[2]Copyright (C) 1986, 1987, 1990, 1991 Colin Kelley, Thomas Williams.

also **[27]**. The factor-circulant matrix generalizes the notion of circulant matrix ($\alpha = 1$) and skew-circulant matrix ($\alpha = -1$) **[10]**. Circulant matrices have interesting and useful properties (**[10]**, **[6]**) that are shared by factor-circulant matrices, e.g., for an arbitrary fixed n and α the factor-circulant matrices form a ring under matrix addition and multiplication.

In **[36]**, the name "α-hyperbolic matrix" was applied to the *transpose* of $H_n^\alpha(x)$, rather than to $H_n^\alpha(x)$ itself. The resulting matrix, the result of multiplying the elements *above* the main diagonal in a circulant matrix by α was described in **[36]** as *factor-circulant*. Our current terminology seems more appropriate in view of the notation used in **[28]** and the notation for skew-circulant matrices in **[10]**. It follows from what was shown in **[36]**, that the α-hyperbolic matrix $H_n^\alpha(x)$ satisfies the identities

$$|H_n^\alpha(x)| = 1, \qquad n \geq 1, \tag{10}$$

and

$$H_n^\alpha(x) \cdot H_n^\alpha(y) = H_n^\alpha(x+y), \qquad n \geq 1, \tag{11}$$

for all real (or complex) x and y, where $|H_n^\alpha(x)|$ is the determinant of the matrix $H_n^\alpha(x)$, and where $\cdot$ denotes matrix multiplication. We will see shortly how these results follow from a study of differential equations in matrix form. Identities (10) and (11) reduce to identities (2) and (3) when $n = 2$ and $\alpha = -1$. When $n = 1$, (11) reduces to the exponential identity

$$e^{\alpha x} e^{\alpha y} = e^{\alpha(x+y)}.$$

6. A Connection to the Binomial Theorem

For $n > 0$ and $\alpha = 0$ (with the usual convention that $0^0 = 1$), the α-hyperbolic matrix $H_n^\alpha(x)$ is upper triangular,

$$H_n^0(x) = \begin{bmatrix} 1 & x & \frac{x^2}{2!} & \cdots & \frac{x^{(n-1)}}{(n-1)!} \\ 0 & 1 & x & \cdots & \frac{x^{(n-2)}}{(n-2)!} \\ 0 & 0 & 1 & \cdots & \frac{x^{(n-3)}}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{12}$$

from which identity (10) is obvious, and identity (11) is a matrix form of the binomial theorem; see **[36]** and **[17]**. This can be seen by considering

$$\begin{bmatrix} 1 & x & \frac{x^2}{2!} & \cdots \\ 0 & 1 & x & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} 1 & y & \frac{y^2}{2!} & \cdots \\ 0 & 1 & y & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x+y & \frac{(x+y)^2}{2!} & \cdots \\ 0 & 1 & x+y & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Every entry above the diagonal in the right-hand matrix is of the form $(x+y)^k/k!$, and it is obtained by multiplying a row of the first matrix on the left by a column of the second matrix on the left. This product is seen to contain exactly those terms that arise in the binomial expansion of $(x+y)^k/k!$. Thus we have what amounts to a link between the binomial theorem and the theory of ordinary differential equations; this has been emphasized in **[34]** and **[37]**. The binomal theorem has a long history **[7]** and is found throughout the mathematical literature mainly in treatises on combinatorial analysis, statistics, and number theory. In literature on differential equations, however, it usually appears only as a tool; see, e.g., **[31]**, p. 404.

7. Differential Equations in Matrix Form

H_n^α is the unique solution of the matrix differential equation

$$M' = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \alpha & 0 & 0 & \dots & 0 \end{bmatrix} M$$

that satisfies $M(0) = I$. That it is a solution is easy to verify. Its uniqueness follows from standard results in the theory of systems of differential equations; see, e.g., **[5]**, Ch. 3. In fact, if we write

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \alpha & 0 & 0 & \dots & 0 \end{bmatrix},$$

we find that

$$H_n^\alpha(x) = \exp(Ax),$$

with the usual notation for the exponential of a matrix **[5**, §3.11**]**. By the usual theory, the Wronskian of such a matrix is constant so we find that $\det H_n^\alpha(x) = \det H_n^\alpha(0) = 1$, i.e., the equation (10). We may also recover (11) with this formulation: For each fixed y, both sides of (11) are matrix solutions of the system $M' = AM$, satisfying $M(0) = F_n^\alpha(y)$. Hence, by uniqueness, they are identical.

It is of interest also to remark that

$$A^n = \alpha I. \tag{13}$$

One way to see this is by direct computation. (Taking successive powers of A causes an upward shift in both the 1's and the α's.) It is also a consequence of the Cayley-Hamilton theorem, according to which A satisfies its own characteristic equation $\det(\lambda I - A) = 0$; expanding by the first column, $\det(\lambda I - A) = \lambda^n + (-1)^{n-1}(-\alpha)(-1)^{n-1} = \lambda^n - \alpha$.

Equation (13) shows that H_n^α satisfies the n-th order matrix differential equation

$$M^{(n)} = \alpha M$$

analogous to the nth-order scalar differential equation (5) satisfied by $F_{n,r}^\alpha(x)$. In fact

H_n^α is the unique matrix solution of the system

$$M^{(n)} = \alpha M, \qquad M^{(k)}(0) = A^k, \qquad k = 0, 1, \ldots, n-1.$$

Another way to express the relationship between $H_n^\alpha(x)$ and A, motivated by the special cases in **[14]**, pp. 337–339, is

$$H_n^\alpha(x) = \sum_{k=0}^{n-1} F_{n,k}^\alpha(x) A^k.$$

We remark that in the case $\alpha = 0$, the matrix $H_n^0(x)$ given by (12), has the familiar form **[5]** of the matrix e^{Ax}, when

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

8. Miscellaneous Results

There are simple formulas for the Laplace transforms of the generalized α-hyperbolic functions. The Laplace transform of $F_{n,r}^\alpha(at)$, $0 \le r < n$, $n = 1, 2, \ldots$, where a and α are real constants, is

$$\int_0^\infty e^{-st} F_{n,r}^\alpha(at)\, dt = \frac{s^{n-r-1} a^r}{s^n + \alpha a_n}. \tag{14}$$

Perhaps the simplest way to see this is to use the differential equations (5) and the well-known relations for Laplace transforms of derivatives of functions. It should be mentioned that the generalized 1-hyperbolic functions are related to the Mittag-Leffler function

$$E_\gamma(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\gamma k + 1)},$$

which is important in the theory of entire functions. The relation is $F_{n,0}^1(x) = E_n(x^n)$.

9. The Fourier Matrix

The Fourier matrix $\mathscr{F}_n$ is defined by

$$\mathscr{F}_n = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n & \omega_n^2 & \ldots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \ldots & \omega_n^{2(n-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \ldots & \omega_n^{(n-1)^2} \end{bmatrix} \tag{15}$$

where $\omega_n = e^{2\pi i/n}$ is a primitive nth root of unity. This is the matrix that arises in connection with the discrete Fourier transform; see, e.g., **[32]** and **[38]**. $\mathscr{F}_n$ connects a column vector of complex numbers $\mathbf{y}=[y_0, y_1, \dots, y_{n-1}]^T$, with its "Fourier coefficients" $\mathbf{c}=[c_0, c_1, \dots, c_{n-1}]^T$ by the equations

$$\mathbf{y}=\mathscr{F}_n\mathbf{c}, \qquad \mathbf{c}=\mathscr{F}_n^{-1}\mathbf{y}. \tag{16}$$

It is a fundamental fact, easily proved by using $1+\omega_n+\omega_n^2+\dots+\omega_n^{n-1}=0$, that $\mathscr{F}_n^{-1}=n^{-1}\overline{\mathscr{F}}_n$, where the bar denotes complex conjugate. There are at least two ways in which the Fourier matrix is connected with the generalized hyperbolic functions. In the first place, equation (6) holds, where $\sqrt[n]{\alpha}$ is an arbitrarily specified nth root of α. Since the nth roots of α are of the form

$$\sqrt[n]{\alpha},\ \omega_n\sqrt[n]{\alpha},\dots,\ \omega_n^{n-1}\sqrt[n]{\alpha},$$

(6) is actually a set of n equations, which in matrix form may be written:

$$\begin{pmatrix} e^{\sqrt[n]{\alpha}x} \\ e^{\omega_n\sqrt[n]{\alpha}x} \\ \dots \\ e^{\omega_n^{n-1}\sqrt[n]{\alpha}x} \end{pmatrix} = \mathscr{F}_n \begin{pmatrix} F^{\alpha}_{n,0}(x) \\ \sqrt[n]{\alpha}F^{\alpha}_{n,1}(x) \\ \dots \\ \left(\sqrt[n]{\alpha}\right)^{n-1}F^{\alpha}_{n,n-1}(x) \end{pmatrix}$$

Using $\mathscr{F}_n^{-1}=n^{-1}\overline{\mathscr{F}}_n$, we easily invert (6) to get (7). Battioni **[2]** gives the relations (8) between functions of orders 4 and 2, a special case of the more general result

$$F^1_{2m,r}(x)=\left[F^1_{m,r}(x)+F^{-1}_{m,r}(x)\right]/2, \quad r=0,1,\dots,m-1, \tag{17}$$

$$F^1_{2m,r+m}(x)=\left[F^1_{m,r}(x)-F^{-1}_{m,r}(x)\right]/2, \quad r=0,1,\dots,m-1, \tag{18}$$

given in a different notation in **[13]**, (33), p. 216. Formulas (17) and (18), which express generalized hyperbolic functions of even order in terms of those of half this order, can be proved in several ways. One way is directly from the series definitions. Another is to show that the functions on both sides of (17) and (18) satisfy the differential equation

$$f^{(2m)}(x)=f(x), \tag{19}$$

while both sides of (17) satisfy the initial conditions

$$f^{(k)}(0)=\begin{cases}0, & k\neq r, \quad 0\le k\le n-1\\ 1, & k=r\end{cases}$$

and both sides of (18) satisfy the conditions

$$f^{(k)}(0)=0, \qquad 0\le k\le n-1.$$

It is instructive to see how the proof of (17) and (18) can be deduced from precisely the same technique as is used in explaining the "fast Fourier transform" or FFT (**[32]**, **[33]**). (The aim of the FFT is to perform the multiplications in (16) quickly, when n is large, by halving n repeatedly. We illustrate one such step, with $n=2m$. The process can be repeated if n is a power of 2.) We have from (7),

$$F^1_{2m,r}(x)=\frac{1}{2m}\sum_{k=0}^{2m-1}\omega_{2m}^{-rk}\exp\left[\omega_{2m}^k x\right] \tag{20}$$

or, in terms of the Fourier matrix,

$$\begin{bmatrix} F^1_{2m,0}(x) \\ F^1_{2m,1}(x) \\ \cdots \\ F^1_{2m,2m-1}(x) \end{bmatrix} = \frac{1}{2m}\overline{\mathscr{F}_{2m}}\begin{bmatrix} e^x \\ e^{\omega_{2m}x} \\ \cdots \\ e^{\omega_{2m}^{2m-1}x} \end{bmatrix}.$$

The idea behind the fast Fourier transform is that the matrix multiplication in the equation

$$\begin{bmatrix} y_0 \\ y_1 \\ \cdots \\ y_{2m-1} \end{bmatrix} = \overline{\mathscr{F}_{2m}}\begin{bmatrix} c_0 \\ c_1 \\ \cdots \\ c_{2m-1} \end{bmatrix},$$

can be achieved in slightly more than half as many steps as would be needed in direct multiplication by splitting the vector on the right-hand side into even and odd components, multiplying these *half-size* vectors

$$\mathbf{c}_{even} = [c_0, c_2, \ldots, c_{2m-2}]^T, \qquad \mathbf{c}_{odd} = [c_1, c_3, \ldots, c_{2m-1}]^T,$$

by $\mathscr{F}_m$ and putting the resulting vectors

$$\mathbf{u} = [u_0, \ldots, u_{m-1}]^T = \overline{\mathscr{F}_m}\mathbf{c}_{even}, \qquad \mathbf{v} = [v_0, \ldots, v_{m-1}]^T = \overline{\mathscr{F}_m}\mathbf{c}_{odd}$$

back together by taking

$$y_j = u_j + \omega_{2m}^{-j} v_j, \qquad j = 0, 1, \ldots, m-1,$$
$$y_{j+m} = u_j - \omega_{2m}^{-j} v_j, \qquad j = 0, 1, \ldots, m-1.$$

In the present situation, we have

$$\mathbf{c}_{even} = \left[e^x, e^{\omega_{2m}^2 x}, \ldots, e^{\omega_{2m}^{2m-2} x}\right]^T, \qquad \mathbf{c}_{odd} = \left[e^{\omega_{2m}x}, e^{\omega_{2m}^3 x}, \ldots, e^{\omega_{2m}^{2m-1} x}\right]^T.$$

Then, using $\omega_{2m}^2 = \omega_m$, we see that (7), with $\alpha = 1, \sqrt[n]{\alpha} = 1$, and $\alpha = -1, \sqrt[n]{\alpha} = \omega_{2m}$ respectively, gives

$$\mathbf{u} = m\begin{bmatrix} F^1_{m,0}(x) \\ F^1_{m,1}(x) \\ \cdots \\ F^1_{m,m-1}(x) \end{bmatrix},$$

and

$$\mathbf{v} = m\begin{bmatrix} F^{-1}_{m,0}(x) \\ \omega_{2m} F^{-1}_{m,1}(x) \\ \cdots \\ \omega_{2m}^{m-1} F^{-1}_{m,m-1}(x) \end{bmatrix}.$$

Using this we get (17) and (18).

10. Graphs and Roots

Here we confine ourselves to the case $\alpha = \pm 1$. See FIGURE 1 for the graphs in the case $\alpha = 1, n = 3$. Battioni [2] provides graphs in a few cases for small values of x. Of course certain results are immediate. From the power series, when n is even, $F_{n,r}(x)$ is even or odd according as r is even or odd. In case $\alpha = -1$, $F_{n,r}$ is a periodic function of x for $n = 2$ but is known not to be periodic for larger values of n [**13**, p. 216]. This can be seen (also for $\alpha = 1$) by using Floquet's theorem [**3**, p. 325]. Of course, it may happen that some particular combination of higher-order functions is periodic; for example:

$$F_{4,0}^{1}(x) - F_{1,2}^{1}(x) = \cos x.$$

The exponential function has no roots. The circular (hyperbolic) functions have only real (purely imaginary) roots. What about the higher-order functions? $F_{n,r}^{-1}(x)$ has a r-fold real root at the origin, but the question of how many other real roots there are is more difficult. Results of Mikusiński [**22**] show that each such function has infinitely many positive roots and that the smallest positive root x_1 satisfies

$$\frac{(r+n-1)!}{(r-1)!} \leq x_1^n \leq \frac{2(r+n-1)!}{(r-1)!}.$$

The graphs provided by Battioni [**2**] indicate that the lower bound is quite sharp. A deeper study of the complex roots has been undertaken by H. Alzer [**1**] who provides two new proofs of a result of G. P. Meyer that $F_{n,r}^{1}(z)$ has infinitely many roots. Generalizing results of G. Pólya and of A. Wiman, Alzer shows that all the roots of $F_{n,r}^{1}(z)$ lie on n rays starting at 0 and passing through the roots of $z^n = -1$. He shows also that the nonzero roots are simple.

FIGURE 1 would seem to indicate relations such as equality of $F_{3,0}^{1}(x)$ and $F_{3,1}^{1}(x)$ at the roots of $F_{3,2}^{1}(x)$. But this is an illusion, based on the fact that e^x is negligible for even moderate negative values of x.

11. History

The history of the α-hyperbolic functions is a fascinating tale of discovery and rediscovery. Most of the history deals with just the cases $\alpha = \pm 1$. The paper by Kaufman [**19**] reveals that these functions were dealt with by Vincenzo Riccati [**25**], son of the better-known Jacopo [**23**] as early as 1757. (Note that according to Katz [**18**] the usual trigonometric functions did not enter calculus until about 1739 and then as a result of Euler's efforts to solve linear differential equations.) The generalized functions also appear in the work of H. Wronski in 1811 and, according to [**19**], are the subject of a chapter in his book [**41**]. Wronski's name is a familiar one to students of differential equations, but only through giving his name to the "Wronskian." See the accounts given in [**12**], [**11**, pp. 57–59] as well as the remark in [**29**, p. 78n] for further information on this fascinating character. The first journal article on the generalized functions and hence the first account to be still reasonably available [**24**] appeared in 1827 in the second volume of Crelle's journal. Kaufman cites more than two dozen 19th-century references to the functions and a similar number in the first half of the 20th century. Some of these references are to rediscoveries. Though it is hard to quantify such an opinion, it seems to us that, compared to similar bibliographies on other topics, a large proportion of the references are to obscure sources, though many (such as [39]) are easily available. One might expect that, with the passage of time, these functions would find their way into standard textbooks and reference books. For the most part this does not seem to have been the case. A notable exception is the five pages in Chapter 18 in the last [13] of the three volumes arising from the Bateman Manuscript Project at the California Institute of Technology in the late 1940s. Nevertheless, the functions have continued to be rediscovered, e.g., in 1969 by Battioni [2], in 1978 by Ricci [26], and in 1982 by one of the present authors [35]. They were discussed in 1987 by Kittappa [20], in 1988 by Good [15], and in 1989 by Coonce, Strachan, and Wiest [8]. The idea of using a general α to unify the special cases $\alpha = \pm 1$ was used by the second author [36] in 1984.

It is interesting to quote some of Davis' remarks on circulants in the Preface to [10]:

> ...The theory of circulants is a relatively easy one. Practically every matrix-theoretic question for circulants may be resolved in "closed form." Thus the circulants constitute a nontrivial but simple set of objects that the reader may use to practice, and ultimately deepen, a knowledge of matrix theory.
>
> Writers on matrix theory appear to have given circulants short shrift, so that basic facts are rediscovered over and over again. ...

It seems that, *mutatis mutandis*, these remarks apply also to the generalized hyperbolic and circular functions.

The first-named author first came across these functions in reviewing Battioni's paper [2] in 1972 (see *Math. Rev.* 45 (1973), #599) and was unaware at that time of the long history of these functions. He was aware of it by the time he reviewed Ricci's paper [26] in 1980 (see *Zbl. für Math.* 423.33008). The second-named author published a note [35] on these functions in 1982 before some of the references given here were drawn to his attention by the first-named author. We hope that this account will help to make these functions more widely known and that others will be led to study the deeper properties of these fascinating functions.

Of course, apart from those mentioned in the Introduction, there are kinds of generalizations of the circular and hyperbolic functions that are different from those described in this paper. In this connection we mention references [4] and [21], for example.

Acknowledgements. We are grateful to several referees for corrections and constructive comments.

REFERENCES

1. H. Alzer, *Die Nullstellen der Hyperbelfunktionen hoeherer Ordnung*, Dissertation, Rheinisch Friedrich-Wilhelms Universität, Bonn, 1983.
2. G. Battioni, Su una generalizzazione delle funzioni iperboliche e delle funzioni circolari. *Riv. Mat. Univ. Parma* (2) 10 (1969), 39–48.
3. G. Birkhoff and G.-C. Rota, *Ordinary Differential Equations*, 3rd edition, John Wiley & Sons, Inc., New York, 1978.
4. B. A. Bondarenko, Generalized hyperbolic functions. *Dokl. Akad. Nauk UzSSR*, 1978, 10–13.
5. M. Braun, *Differential Equations and their Applications*, 3rd edition, Springer-Verlag, New York, 1983.
6. R. Chalkley, Circulant matrices and algebraic equations, this MAGAZINE 48 (1975), 73–80.
7. J. L. Coolidge, The story of the binomial theorem, *Amer. Math. Monthly* 56 (1949), 147–157.
8. H. B. Coonce, R. A. Strachan and M. Wiest, True trigonometry, talk given at MAA Fall Meeting, North Dakota State University, October 27–28, 1989.
9. H. B. Coonce, Permutable polynomials for several variables, *Aequationes Math.* 44 (1992), 292–303.

10. P. J. Davis, *Circulant Matrices*, John Wiley & Sons, Inc., New York, 1979.
11. P. J. Davis and R. Hersh, *The Mathematical Experience*, Birkhäuser, Basel-Boston-Stuttgart, 1981.
12. J. Dobrzychi, Józef Maria Hoëne-Wroński, *Dictionary of Scientific Biography*, 15, pp. 225–226, Scribner's, New York, 1978.
13. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. 3, McGraw-Hill, New York, 1955.
14. A. E. Fekete, *Real Linear Algebra*, Marcel Dekker, Inc., New York and Basel, 1985.
15. I. J. Good, A simple generalization of analytic function theory, *Expo. Math.* 6 (1988), 289–311.
16. E. Atlee Jackson, Perspectives on Nonlinear Dynamics, Vol. 1, Cambridge University Press, 1990.
17. D. Kalman and A. Ungar, Combinatorial and functional identities in one-parameter matrices. *Amer. Math. Monthly* 94 (1987), 21–35.
18. V. J. Katz, The calculus of the trigonometric functions, *Historia Math.* 14 (1987), 311–324.
19. H. Kaufman, A bibliographical note on the higher sine functions. *Scripta Math.* 28 (1967), 29–36.
20. R. Kittappa, A generalization of the rotation matrix and related results. *Lin. Alg. Appl.* 92 (1987), 251–258.
21. A. I. Markushevich, *The remarkable sine functions*, trans. by Scripta Technica, trans. edited by L. Ehrenpreis, American Elsevier, New York, 1966.
22. J. G. Mikusiński, *Ann. Soc. Polon. Math.* 21 (1948), 46–51.
23. A. Natucci, Vincenzo Riccati, *Dictionary of Scientific Biography*, 11, pp. 401–402, Scribner's, New York, 1975.
24. L. Olivier, Bemerkungen über eine Art von Functionen, welche ähnliche Eigenschaften haben, wie die Cosinus und Sinus, *J. Reine Angew. Math.* 2 (1827), 243–251.
25. V. Riccati, Opusculum sextum: Epistolae tres, in quibus aequationes aliquae differentiales evolvuntur per series, *Opusculorum ad res Physicas, & Mathematicas pertinentium*, Vol. 3, Bologna, 1757.
26. P. E. Ricci, Le funzioni pseudo-iperboliche e pseudo-trigonometriche, *Publ. Instit. Mat. Appl. Fac. Ingr. Univ. Stud. Roma*, quaderno no. 12 (1978), 37–49.
27. J. C. Ruiz-Claeyssen, M. Davila and T. Tsukazan, Factor block circulant and periodic solutions of undamped matrix differential equations, *Mat. Apl. Comput.* 3 (1983), No. 1.
28. J. C. Ruiz-Claeyssen and L. A. dos Santos Leal, Diagonalization and spectral decomposition of factor block circulant matrices, *Lin. Alg. Appl.* 99 (1988), 41–61.
29. G. F. Simmons, *Differential Equations with Applications and Historical Notes*, McGraw-Hill, New York, 1972.
30. L. J. Slater, *Hypergeometric Functions*, Cambridge University Press, Cambridge, UK, 1960.
31. M. R. Spiegel, *Applied Differential Equations*, Prentice-Hall, Englewood, NJ, 1961.
32. G. Strang, *Linear Algebra and its Applications*, 3rd edition, Harcourt Brace Jovanovich, New York, 1988.
33. G. Strang, *An Introduction to Applied Mathematics*, Wellesley-Cambridge Press, Cambridge, MA, 1986.
34. A. Ungar, Addition theorems for solutions to linear homogeneous constant coefficient differential equations. *Aequationes Math.* 26 (1983), 104–112.
35. A. Ungar, Generalized hyperbolic functions, *Amer. Math. Monthly* 89 (1982), 688–691.
36. A. Ungar, Higher order alpha-hyperbolic functions. *Indian J. Pure Appl. Math.* 15 (1984), 301–304.
37. A. Ungar, Addition theorems in ordinary differential equations, *Amer. Math. Monthly* 94 (1987), 872–875.
38. J. M. Walker, *Fourier Analysis*, Oxford University Press, New York, 1988.
39. L. E. Ward, Some functions analogous to trigonometric functions, *Amer. Math. Monthly* 34 (1927), 301–303.
40. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd edition, Cambridge University Press, Cambridge, UK, 1944.
41. H. Wronski, *Reforme Absolue du Savoir Humain*, 3 Vols., 1847–48.
42. W. W. Zachary, An inverse scattering formalism for higher-order differential operators. *J. Math. Anal. Appl.* 117 (1986), 449–495.

A Round-Up of Square Problems

DUANE DETEMPLE
SONIA HAROLD
Washington State University
Pullman, WA 99164-3113

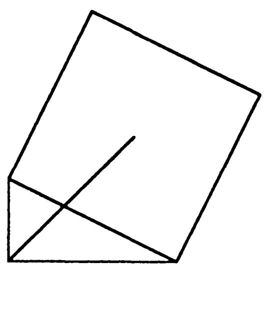
Introduction

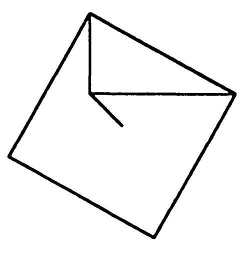
Are squares, as their name suggests, really the boring "nerds" of the geometric world? We think not, and have gathered a number of our favorite problems that we hope show the square to be a fascinating figure. Many of the results deserve to be (and indeed are) theorems, but much of the fun seems to be in presenting the material in the form of problems. You are challenged to find your own solutions, and we hope you won't jump too quickly to the solutions we have provided.

The problems are, at least roughly, divided into sections according to the number of squares involved. Many of the problems are new, at least to us, and others may be familiar to some readers. Even then, some novelty is found in most of the solutions, and in several places we have uncovered unexpected connections among what at first may seem to be unrelated problems. A concluding section provides some sources, though it is not always easy to know who deserves first credit. There should be little harm in rediscovering a neglected gem, and much interest and pleasure to be gained.

Problems About One Square

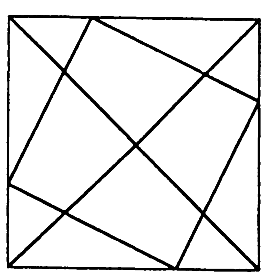
Problem 1. A square is erected, either externally or internally, on the hypotenuse of a right triangle. Show that the line segment from the vertex of the right angle to the center of the square makes 45° angles to the legs of the right triangle.

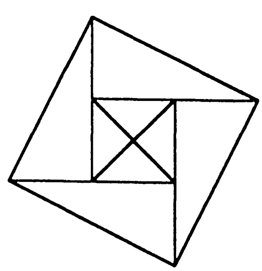




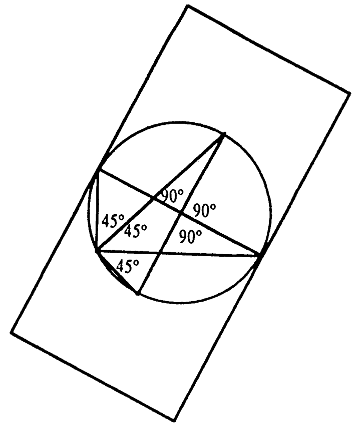
Solution 1. Here are two nice ways to solve this problem.

(a) *via tiling:* Adding congruent copies of the right triangle to the remaining sides of the given square gives us a second square that makes the result visually obvious. The segment through the center of the second square is along a diagonal of the new square, so it bisects the right triangle.

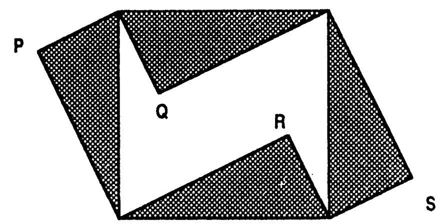




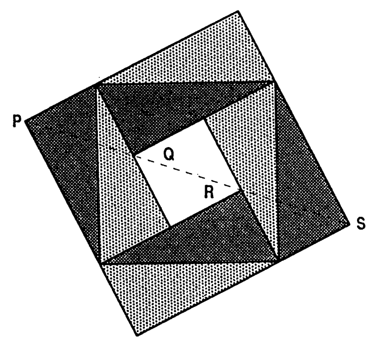
(b) *via the inscribed angle theorem:* Both the internally and externally erected square cases can be shown together. Construct the circle centered on the hypotenuse of the right triangle. The legs of the right triangle and the segments to the centers of the squares intercept 90° arcs on the circle. By the inscribed angle theorem, each inscribed angle has half the measure of the 90° arc, namely 45°.



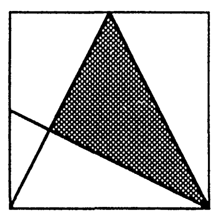
Problem 2. Congruent right triangles are erected to the sides of a square, facing alternately outward and inward as shown. Show that P, Q, R, and S are collinear.



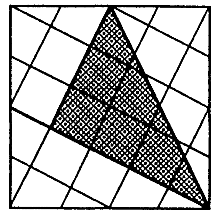
Solution 2. Combining the tilings shown in the solution of Problem 1 reveals that P, Q, R, and S are all on the diagonal of a circumscribed square.



Problem 3. The shaded triangle at the right is formed by drawing segments from corners of the square to the midpoints of opposite sides, as shown. Show that the triangle is a right triangle with sides in the proportion 3 : 4 : 5.

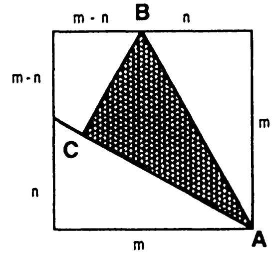


Solution 3. There is an elegant tiling solution, formed by overlapping the given figure in a square grid containing the points $\frac{1}{4}$, $\frac{2}{4}$, and $\frac{3}{4}$ along the edges of the square.



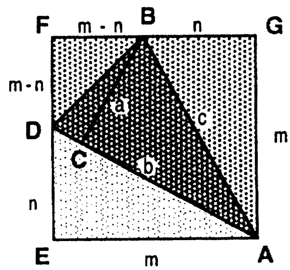
Problem 4. Let m and be positive integers with $m > n$. The shaded right triangle, ΔABC, is constructed in an m by m square as shown. Show that the triangle has sides in the integer proportions

$$(m^2 - n^2): 2mn: (m^2 + n^2).$$



(*Note:* Choosing m and n relatively prime and of opposite parity, it is well known that all primitive Pythagorean triples are of the form $m^2 - n^2$, $2mn$, and $m^2 + n^2$. Thus the construction realizes all of the right triangles with integer sides in appropriately sized squares.)

Solution 4. The square $AEFG$ can be viewed as being dissected into four triangles, as shown. We then obtain the area equation



$$area(AEFG) = area(\Delta ADE) + area(\Delta ABG) + area(\Delta BFD) + area(\Delta ABD).$$

Letting

$$a = BC, b = AC, c = AB = AD = \sqrt{m^2 + n^2},$$

we see that the area equation becomes

$$m^2 = \frac{mn}{2} + \frac{mn}{2} + \frac{(m-n)^2}{2} + \frac{ac}{2}.$$

Solving for a we find $a = (m^2 - n^2)/c$, from which we learn that

$$b^2 = c^2 - a^2 = \frac{(c^4 - a^2c^2)}{c^2} = \frac{(c^2 - ac)(c^2 + ac)}{c^2}$$

$$= \frac{(m^2 + n^2 - m^2 + n^2)(m^2 + n^2 + m^2 - n^2)}{c^2} = \frac{4m^2n^2}{c^2}.$$

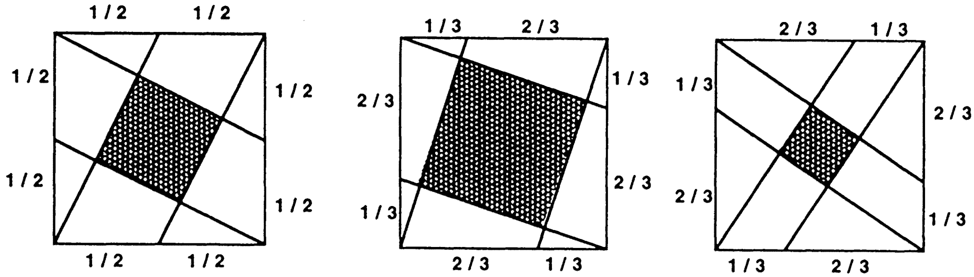
That is, $b = 2mn/c$. Writing $c = c^2/c = (m^2 + n^2)/c$ we see that

$$a = \frac{(m^2 - n^2)}{c}, b = \frac{2mn}{c}, c = \frac{(m^2 + n^2)}{c}.$$

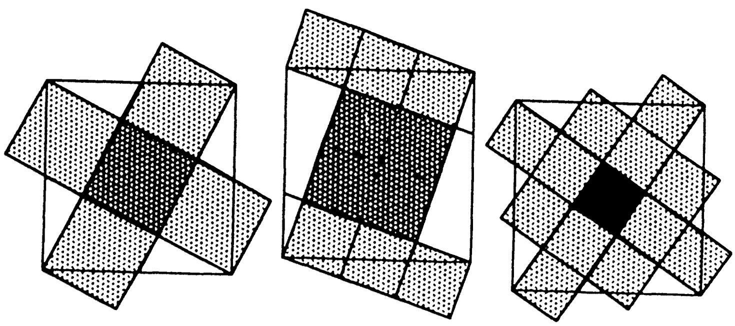
Therefore, $a : b : c = (m^2 - n^2) : 2mn : (m^2 + n^2)$.

Problems About Two Squares

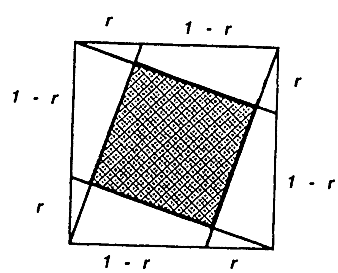
Problem 5. A square is created by connecting each vertex of the unit square to a point on a nonadjacent side, as shown in these three examples. What is the area of each shaded square?



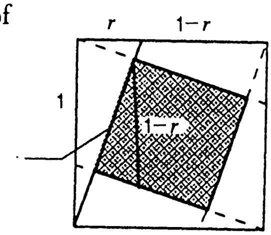
Solution 5. In each case, an inscribed square grid makes the answers readily apparent as shown below. In each case the triangular regions lying in the exterior of the original unit square are paired with a congruent triangle within the unit square that lies outside the shaded square region. The dark-shaded squares are seen to have respective areas $\frac{1}{5}$, $\frac{4}{10} = \frac{2}{5}$, and $\frac{1}{13}$.



Problem 6. Find the area of the shaded square contained within the unit square as shown, where $0 < r < 1$.



Solution 6. A vertical segment drawn from a vertex of the shaded square to the opposite side has length $1 - r$, compared to a length of $\sqrt{1+r^2}$ of a corresponding segment in the unit square. Thus the ratio of similarity is $(1-r)/\sqrt{1+r^2}$, making the area of the shaded square

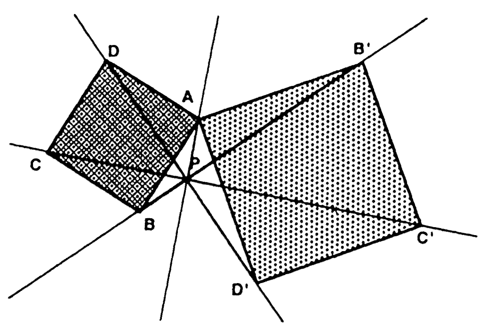


$$A = \frac{(1-r)^2}{1+r^2}.$$

For example, if $r = \frac{2}{3}$, then $A = \dfrac{\left(1-\frac{2}{3}\right)^2}{1+\left(\frac{2}{3}\right)^2} = \dfrac{\frac{1}{9}}{\frac{13}{9}} = \dfrac{1}{13}$

in agreement with the dissection solution shown in Problem 5.

Problem 7. Let the squares $ABCD$ and $AB'C'D'$ share a vertex at A, where both squares are labeled clockwise.

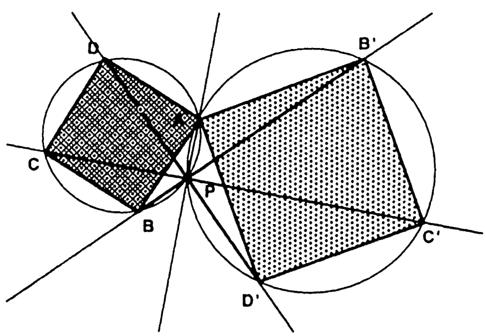


(a) Show that the segments BB' and DD' are the same length and lie on perpendicular lines.

(b) Let P be the point at which the perpendicular lines BB' and DD' intersect. Show that the line CC' also passes through P, and is an angle bisector.

(c) Show that the line AP is perpendicular to line CC'.

Solution 7. (a) A 90° rotation about point A transforms $\Delta ABB'$ onto $\Delta ADD'$, showing that the triangles are congruent. In particular, $BB' = DD'$ and are contained in lines that cross at 90°.

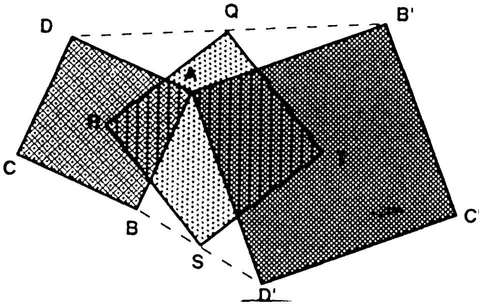


(b) Draw the circumscribing circles of each of the squares. These circles intersect at A and P so (cf. Solution 1(b)) by the inscribed angle theorem we see that PC and PC' are each angle bisectors of the right angles at P.

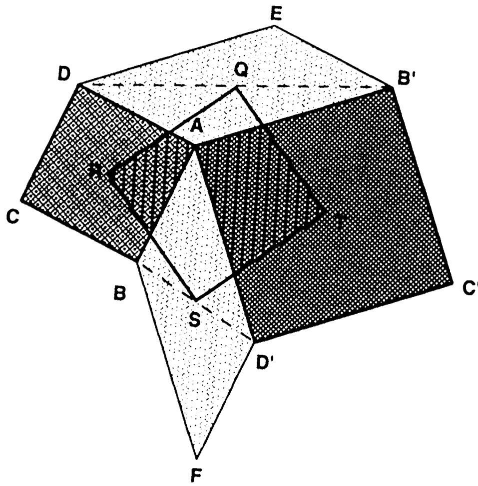
(c) Since the rays PA and PC intercept diametrically opposite points A and C of the circumscribing circle, $\angle APC$ is a right angle.

Remark. An alternate proof of parts (b) and (c) can be based on the results of part (a) and Problem 1.

Problem 8. Let squares $ABCD$ and $AB'C'D'$ share a vertex (as in Problem 7). Show that the midpoints, Q and S, of the segments $B'D$ and BD' together with the centers R and T of the squares form another square, $QRST$.

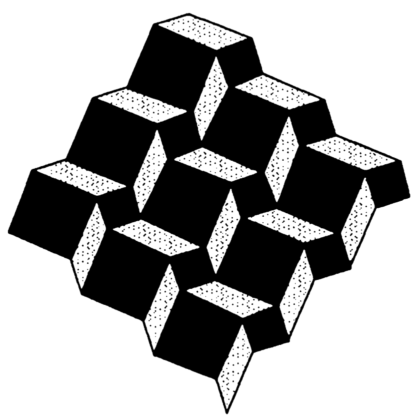


Solution 8. A pair of congruent parallelograms, $AB'ED$ and $BFD'A$ have Q and S as their respective centers. Since a 90° rotation about R transforms $AB'ED$ onto $BFD'A$ we see that RS and RQ are congruent segments meeting at 90°. Similarly, QT and ST are congruent and orthogonal, so it follows easily that $QRST$ is a square.



Remark 1: This result is sometimes known as the *Finsler-Hadwiger* theorem. It will be convenient later to refer to $QRST$ as the *Finsler-Hadwiger square* determined by the given squares sharing a vertex at point A. Note that the entire configuration is uniquely determined by the three points A, R, and T.

Remark 2: A visualization of the generation of the Finsler-Hadwiger squares is provided by tiling the plane with the octagon shown above. The centers of the parallelograms and squares are seen to form a square grid. A second square grid, of twice the linear size, is formed by the translates of the square $CEC'F$. This same tiling can also be used to visualize the results of Problem 7.



Remark 3: The result of Problem 8 is actually a special case of a more general theorem that is elementary yet of interest.

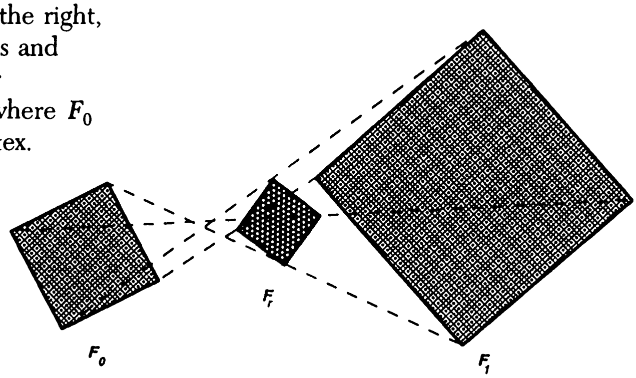
THEOREM. *Let F_0 and F_1 denote two directly similar figures in the plane, where $P_1 \in F_1$ corresponds to $P_0 \in F_0$ under the given similarity. Let $r \in (0, 1)$, and define $F_r = \{(1-r)P_0 + rP_1 \colon P_0 \in F_0\}$. Then F_r is also directly similar to F_0.*

Proof. We assume the figures are in the complex plane, so that the similarity has the form $z \mapsto az + b$, where a and b are complex constants with $a \neq 0$. Thus F_0 is mapped to F_r by the map

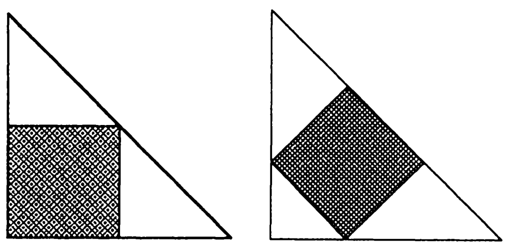
$$\sigma_r(z) = (1-r)z + r(az+b) = (1-r+ra)z + rb,$$

which has the form of a direct similarity transformation. ∎

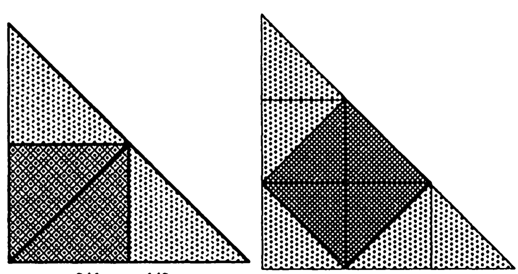
An example of the theorem is illustrated in the diagram at the right, where the figures are squares and $r = \frac{1}{2}$. The Finsler-Hadwiger theorem is the special case where F_0 and F_1 share a common vertex.



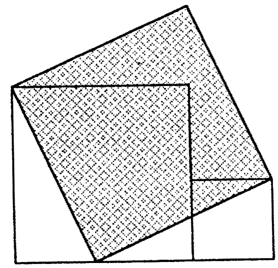
Problem 9. Squares have been inscribed in congruent isosceles right triangles in two different ways. Which square has the larger area?



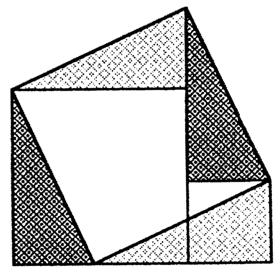
Solution 9. Triangular grids show that the respective areas are $\frac{2}{4}$ and $\frac{4}{9}$. Thus $\frac{1}{18}$, or about $5\frac{1}{2}\%$, more of the triangle's area is covered by the square on the left.



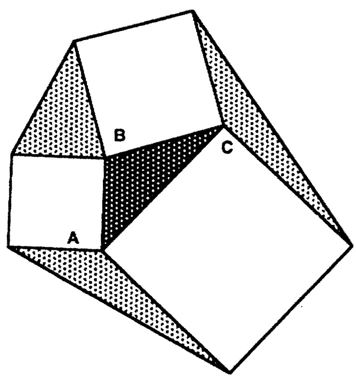
Problems About Three Squares



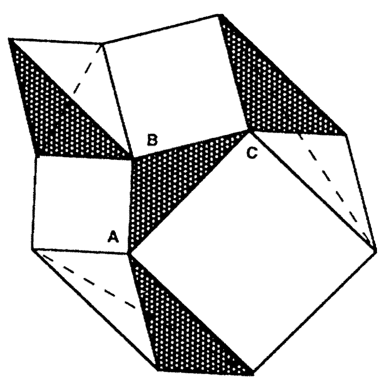
Problem 10. Two side-by-side squares are constructed on a horizontal segment. The upper left-most and right-most vertices are then used as opposite vertices of a tilted larger square. Show that the large square has one vertex on the horizontal segment and another vertex on the extension of the common vertical sides of the small squares. Then compare the areas of the three squares.



Solution 10. The configuration described in the problem statement is a thinly-disguised confirmation of the Pythagorean theorem, which is surely the most famous result about three squares in all of geometry. The dissected figure at the right makes it visually clear that the area of the large square is the sum of areas of the two smaller squares. The dissection is attributed to Tâbit ibn Qorra (826–901), and was rediscovered in 1873 by Henry Perigal.

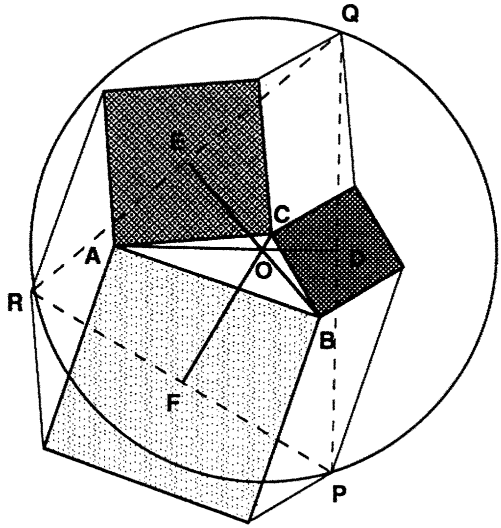


Problem 11. Given any triangle ABC, erect outward facing squares on all three sides. Three additional triangles are then constructed, as shown in the figure. Show that all four triangles have the same area.

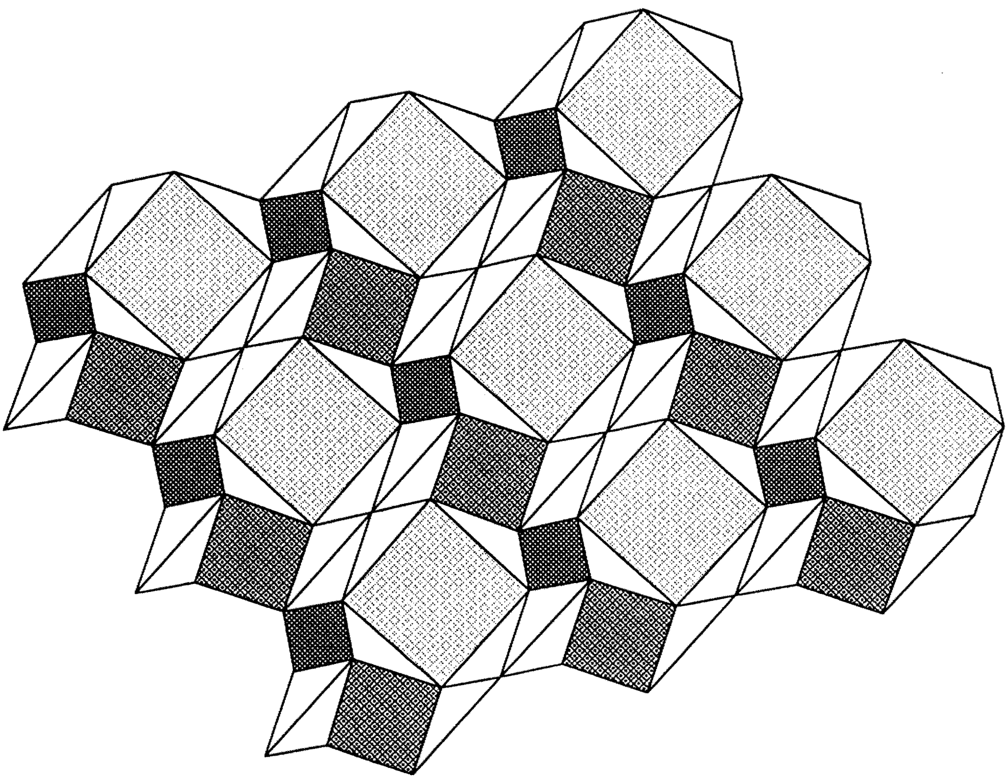


Solution 11. The tiling shown in the solution to Problem 8 provides a simple way to see why the triangles have equal area: extend each outer triangle to a parallelogram. Drawing the opposite diagonal forms triangles that are all congruent to ΔABC, and therefore have areas equal to the original triangle of the problem.

Problem 12. Outward facing squares with centers D, E, and F are erected on the sides of an arbitrary triangle ABC. Next, parallelograms are constructed as shown, determining P, Q and R. Show that the segments AD, BE, and CF are concurrent at a point O that is the center of the circumscribed circle of ΔPQR.



Solution 12. The result is evident in the beautiful tiling shown below. For example, we easily see that 90° and $-90°$ rotations about D will take the point A to P and Q, respectively. Thus AD is the perpendicular bisector of PQ.



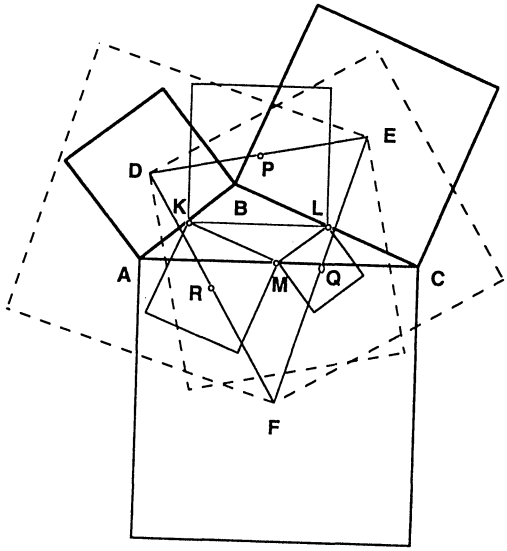
Problems About Four or More Squares

Problem 13. Let squares be erected externally on the sides of a triangle ABC, with centers D, E, and F.

(a) Show that the midpoints K, L, and M of the sides of ΔABC coincide with the centers of the squares erected internally on the sides of triangle DEF.

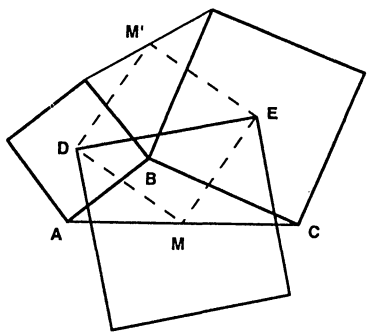
(b) Show that the centers P, Q, and R of the squares erected externally on the sides of ΔKLM coincide with the midpoints of the sides of ΔDEF.

The properties also hold if internally and externally are interchanged.

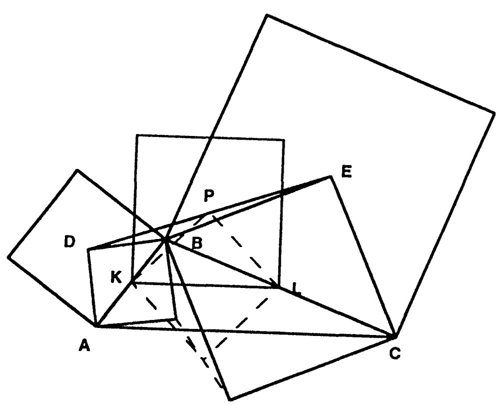


Solution 13.

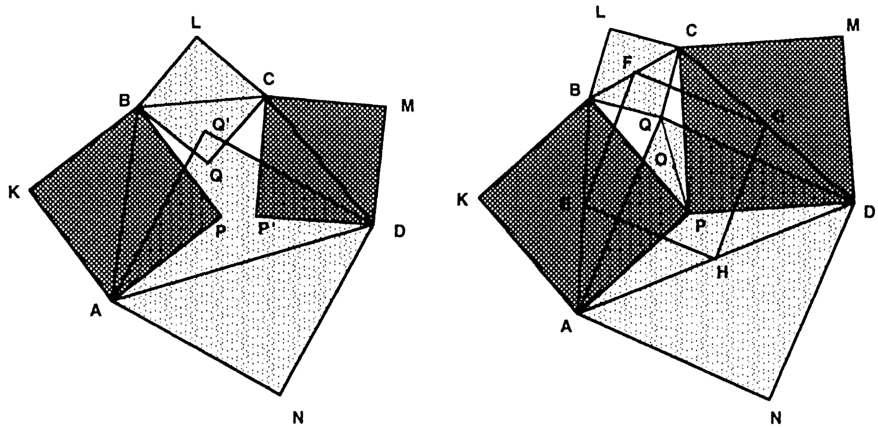
(a) The Finsler-Hadwiger square (shown dashed) determined by the squares centered at D and E (see *Remark 1* in Solution 8) has one vertex at M, the midpoint of side AC. But M is also seen to be the center of the square with side DE.



(b) Construct squares with diagonals AB and BC. The Finsler-Hadwiger square corresponding to the squares whose diagonals are AB and BC has P, the midpoint of DE, as a vertex. Clearly P is also the center of the externally erected square on side KL.



Problem 14. Construct squares whose diagonals are the sides of a quadrilateral $ABCD$. Let K, L, M, and N denote the external vertices, and P, Q, P', and Q' the internal vertices, of the squares, as shown at the left below.

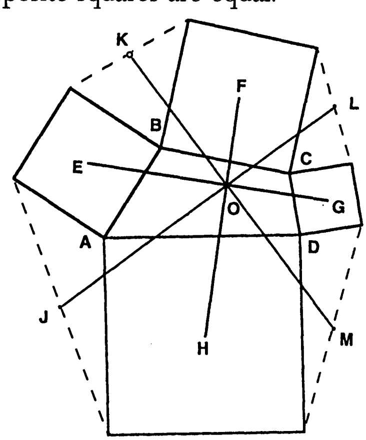


(a) Show that $P = P'$ if and only if $Q = Q'$, as shown above on the right.
(b) In the case that $P = P'$ (and $Q = Q'$), show that:

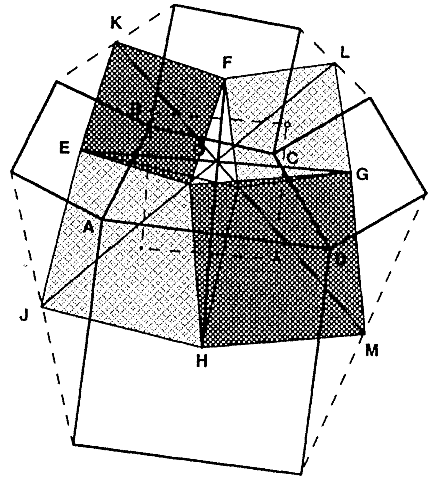
- the midpoints of the sides of $ABCD$ form a square, $EFGH$;
- the center, O, of square $EFGH$ is also the midpoint of segment PQ;
- the sum of the areas of the two squares sharing vertex P is equal to the sum of the areas of the two squares sharing vertex Q.

Solution 14. Suppose that $P = P'$. Then the squares with diagonals AB and CD generate the Finsler-Hadwiger square $EFGH$, which has its vertices at the midpoints of the sides of $ABCD$. By Remark 1 following Solution 8, there is a unique square centered at F which, together with the square $AQ'DN$, generates $EFGH$ as their corresponding Finsler-Hadwiger square. But this square, centered at F, has diagonal BC, so $Q = Q'$. By the result in Problem 13(a) (which is *Neuberg's theorem*), applied to ΔBQP, we deduce that the center O of the square with side EF is at the midpoint of QP. By the result of Problem 7, the diagonals AC and BD lie on perpendicular lines; thus, as easily follows from the Pythagorean theorem, $AB^2 + CD^2 = BC^2 + DA^2$. This equation shows that the sums of the areas of opposite squares are equal.

Problem 15. Squares are erected externally on the sides of quadrilateral $ABCD$, with centers E, F, G and H. Show that the segments EG and FH are congruent and lie on perpendicular lines. Similarly, if J, K, L, and M are the midpoints of the dashed segments shown, prove that JL and KM are congruent segments that lie on perpendicular lines, with the length of these segments $\sqrt{2}$ times the length of EG and FH. Moreover, show that all four lines are concurrent, intersecting at point O at 45° angles.



Solution 15. The configurations discovered in some of the preceding problems provide the keys. By Problem 13, the squares with diagonals EF and GH have a common vertex at the midpoint of AC, as we see in the figure at the right. Similarly, the squares with diagonals FG and EH have a common vertex at the midpoint of BD. Problem 7 showed us that EG and FH are congruent and lie on perpendicular lines, that the same property holds for JL and KM, and that all four lines are concurrent. Moreover, the common length of EF and GH is twice the length of the side of the Finsler-Hadwiger square (shown dashed) formed by the centers of the newly constructed squares. Similarly, the common length of JL and KM is twice the length of the diagonal of the Finsler-Hadwiger square.



Sources and Additional Remarks for Selected Problems

Problem 1 was inspired by a problem of Larry Hoehn [8]. The case of the internally erected square, and the inscribed angle proof, are apparently new. The first case of Problem 5 is attributed to Heinrich Dörrie by Edward Kitchen [9]. Kitchen's article solves the second case with a different tiling than ours, and also discusses a number of similar problems dealing with squares. Problem 7 (b) and (c), in a slightly different form, appeared as the first two parts of a problem of Andrew Cusumano [2]; his references indicate that the problem has reappeared several times beginning in 1919. A solution to the problem in [11] is similar to ours.

The result of Problem 8, which introduced the Finsler-Hadwiger square [4], is proved in [5] in a very different way; [5] also contains a list of references, supplied by Murray Klamkin, related to the Finsler-Hadwiger theorem. The tiling shown in Remark 2 seems to be a new connection to the theorem (in a strange coincidence, almost to the day the tiling was first drawn, the same pattern was seen worn on a tie by comedian Tim Allen in the popular television show *Home Improvement!*). The result of Remark 3 was a rediscovery of what Howard Eves calles the *fundamental theorem of directly similar figures* [3]; the application to the Finsler-Hadwiger theorem seems to be new.

Problem 9 was contributed by James Varnadore [12] as a calendar problem, but the simple dissection proof we have given is new. Problem 11 is due to Bishnu Naraine [10], who gives a trigonometric solution. A letter of Bo Burbank [1] gives the beautiful transformational proof we have reproduced. Problem 12 and the tiling shown in the solution seem to be new. Part (a) of Problem 13 is due to Joseph Neuberg (1840–1926); see [7]. Problem 14 (a) is a variant of the Douglas-Neumann theorem, discovered independently by Jesse Douglas and B. H. Neumann in 1940; see [3] for references. The first part of Problem 15, which shows the congruence and orthogonality of the segments connecting opposite erected squares on a quadrilateral, is the well-known theorem of von Aubel (see [9], for example, for a vector proof). The extensions in Problems 13, 14, and 15 seem to be new, as are the connections to the Finsler-Hadwiger theorem in those problems.

REFERENCES

1. B. Burbank, Letter in Reader Reflections column, *Math. Teacher* 86:8 (1993), 701.
2. A. Cusumano, Problem 817, *Pi Mu Epsilon J*. 9:9 (Fall 1993), 619.
3. H. Eves, Solution of Problem E 521, *Amer. Math. Monthly* 50 (1943), 64.
4. P. Finsler, H. Hadwiger, Einige Relationen im Dreieck, *Comment. Helv.* 10 (1937), 316–326.
5. J. C. Fisher, D. Ruoff, J. Shilleto, Polygons and polynomials, in *The Geometric Vein: The Coxeter Festschrift*, Springer-Verlag, New York, 1981, 321–333.
6. P. J. Kelly, Von Aubel's quadrilateral theorem, this MAGAZINE 49 (1966), 35–37.
7. R. Honsberger, *More Mathematical Morsels*, Mathematical Association of America: Washington DC, 1990.
8. L. Hoehn, Calendar Problem 5, *Math. Teacher* 83:1 (1990), 38.
9. E. Kitchen, Dörrie tiles and related miniatures, this MAGAZINE 67:2 (1994), 128–130.
10. B. Naraine, If Pythagoras had a geoboard..., *Math. Teacher* 86:2 (1993), 137–140, 145–148.
11. *Pi Mu Epsilon J*. 10:1 (1994), 72–73 (solution of Problem 817).
12. J. Varnadore, Calendar problem 18, *Math. Teacher* 84:4 (1991), 291.

NOTES

On Using Flows to Visualize Functions of a Complex Variable

TYRE NEWTON
THOMAS LOFARO[1]
Washington State University
Pullman, WA 99164-3113

Introduction Braden in [2] used Pólya-Latta vector fields [8] to provide a new geometric interpretation of functions of a complex variable. For a given function F of a complex variable z, Pólya and Latta associated with each z in the domain of F the vector whose components are the real and imaginary parts of the complex conjugate $\overline{F(z)}$. This technique yields a divergence-free and curl-free vector field when F is analytic. Moreover, it follows from the Cauchy-Riemann equations that the zeros of multiplicity one of $F(z)$ are saddles for the flow associated with $\overline{F(z)}$. Gluchoff in [4] used this technique to illustrate complex power series.

By further investigating the concept of vector fields associated with $F(z)$, we obtain some interesting computer graphics that give a dynamic interpretation of functions of a complex variable. Moreover, these phase portraits can be interpreted in much the same manner as the Pólya-Latta vector fields of $\overline{F(z)}$. They illustrate both the index of a zero or pole of F as discussed by Braden and the properties of power series as presented by Gluchoff. When applied to series expansions, these graphics give us visual examples of the approximation properties of truncated series and associated limitations. The behavior of analytic functions at infinity is also illustrated. These features are at a level to be appreciated by the undergraduate student when he or she is first introduced to functions of a complex variable.

Preliminary definitions When we look at the Pólya-Latta vector fields as illustrated in [2] we are immediately reminded of the direction fields for 1-dimensional ordinary differential equations as discussed by Boyce and DiPrima [1, p. 35] and 2-dimensional vector fields as discussed by Hale and Koçak [5, p. 179]. For $F = u + iv$, a complex-valued function of a complex variable $z = x + iy$, $(i = \sqrt{-1})$, the equation

$$\dot{z} = F(z) \tag{1}$$

$((\dot{\ }) = d/dt)$ defines a vector field on the domain of F. This is seen by considering the equivalent 2-dimensional system of differential equations

$$\dot{x} = u(x, y), \quad \dot{y} = v(x, y) \tag{1'}$$

that associates with each $z = x + iy$, the vector $\mathbf{V} = (u, v)$, to define the *vector field* of

[1]To whom correspondence should be addressed.

F. The Pólya-Latta vector field considered by Braden and by Gluchoff is the vector field of the complex conjugate function $\overline{F}$.

A function $\Phi(t) = \phi(t) + i\psi(t)$ $(= (\phi(t), \psi(t)))$ that satisfies (1) (or (1′)) for all t defines a curve in the z plane (xy-plane). When arrows are inserted to indicate the direction of flow with increasing time, the result is an *orbit* or *trajectory* of (1) (or (1′)). The collection of orbits yields the *flow* or *phase portrait* of F. We will also refer to (1) or (1′) as *vector fields*.

We should note that if F is analytic at z, then the Existence and Uniqueness Theorem for (1′) assures us that one, and only one, orbit of (1) passes through each z. Furthermore, (1′) tells us that the velocity vector at z is precisely the value of $F(z)$. Thus visualizing the velocity vectors of the flow of F truly depicts the behavior of F over its domain.

Computer graphics Our first example is the function $F(z) = e^z = e^x(\cos y + i \sin y)$. In this case, system (1′) becomes

$$\dot{x} = e^x \cos y, \quad \dot{y} = e^x \sin y. \tag{2}$$

However, since $e^{-x} > 0$ for all x, the phase portrait for this system coincides with that of

$$\dot{x} = \cos y, \quad \dot{y} = \sin y. \tag{2'}$$

Note that multiplying a vector field on the right by a continuous function that does not change sign may change the parameterization of each solution curve, but does not change the geometry of the orbits. The phase portrait of (2′) (or the flow of e^z) is given in FIGURE 1a. Note that the periodicity of e^z with respect to its imaginary part is evident from its flow. In particular, the lines $y = n\pi, n = \pm 1, \pm 2, \ldots$ are invariant, reflecting the fact that these lines are mapped onto the real axis by e^z.

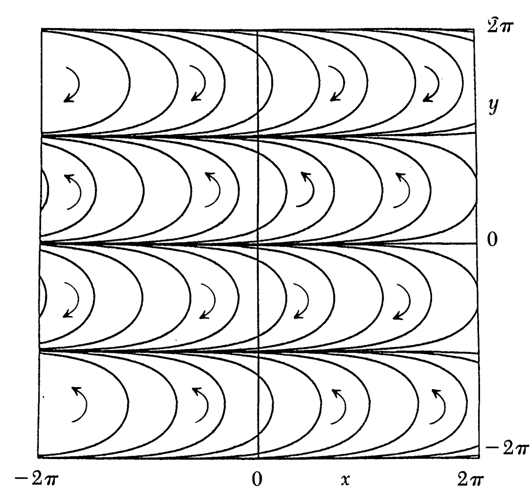


FIGURE 1a
The flow of $\dot{z} = e^z$.

To illustrate the flow at infinity, we make the change of variable $z = 1/w$ so that behavior in a neighborhood of $z = \infty$ in the equation $\dot{z} = e^z$ is mapped to behavior in a neighborhood of $w = 0$ in

$$\dot{w} = -w^2 e^{1/w}.$$

This flow in the xy-plane is shown in FIGURE 1b. In FIGURE 1c we project the flow e^z onto the Riemann sphere $x^2+y^2+(z-1/2)^2=1/4$ [7, p. 61]. Keep in mind that in FIGURE 1c, $\{z:|z|>1\}\cup\{\infty\}$ maps one-to-one onto the upper hemisphere. Note that the geometry of the solutions in FIGURE 1c near infinity is consistent with the geometry in FIGURE 1b. Given the periodicity and regularity observed in FIGURE 1a, the behavior near infinity for the flow of e^z is at first surprising. Recall, however, that on the Riemann sphere, $z=\infty$ is an essential singularity of e^z and thus by Picard's theorem e^z takes on all values (except perhaps one) infinitely often in any neighborhood of infinity. The dynamics illustrated here have been discussed thoroughly by Hockett and Ramamurti [6]. The dynamics near essential singularities of the other entire functions ($\sin z$, $\cos z$, etc.) are not as thoroughly understood.

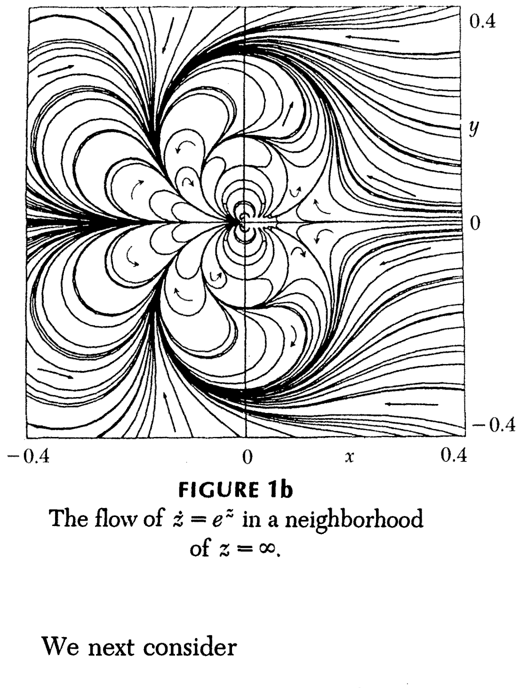


FIGURE 1b
The flow of $\dot{z}=e^z$ in a neighborhood of $z=\infty$.

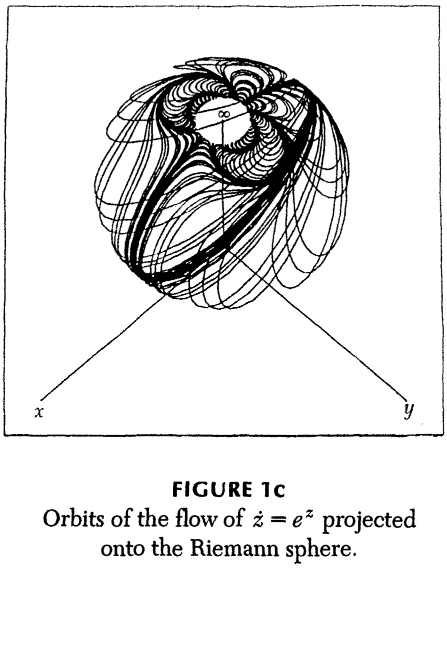


FIGURE 1c
Orbits of the flow of $\dot{z}=e^z$ projected onto the Riemann sphere.

We next consider

$$\mathrm{Log}(1+z)=\frac{1}{2}\ln\left((1+x)^2+y^2\right)+i\,\mathrm{Arctan}\left(\frac{y}{1+x}\right).$$

In this case (1′) becomes

$$\dot{x}=\frac{1}{2}\ln\left((1+x)^2+y^2\right),\quad \dot{y}=\mathrm{Arctan}\left(\frac{y}{1+x}\right) \tag{3}$$

where $-\pi<\mathrm{Arctan}(y/(1+x))\le\pi$. FIGURE 2a shows the phase portrait of system (3) (or the flow of $\mathrm{Log}(1+z)$). $\mathrm{Log}(1+z)$ is plotted so that $z=-1$ is the *branch point*, and the negative real axis for which $z<-1$ is the *branch cut*. The discontinuity of $\mathrm{Log}(1+z)$ along this ray is reflected in FIGURE 2a by the cusps in trajectories intersecting this ray. FIGURE 2b illustrates the flow associated with

$$F(z)=\frac{1}{1+z},$$

the derivative of our previous function. For this function $z=-1$ is not a branch point, but a pole of order 1.

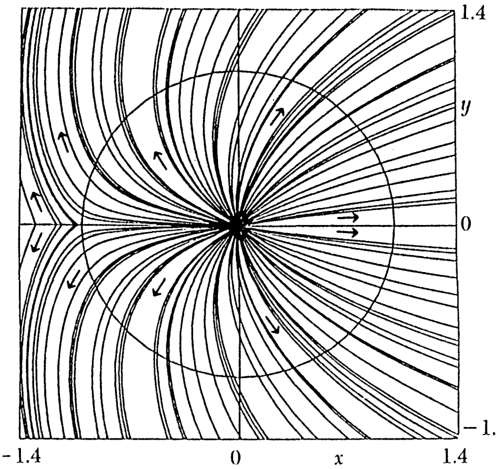


FIGURE 2a
The flow of $\dot{z} = \mathrm{Log}(1+z)$.

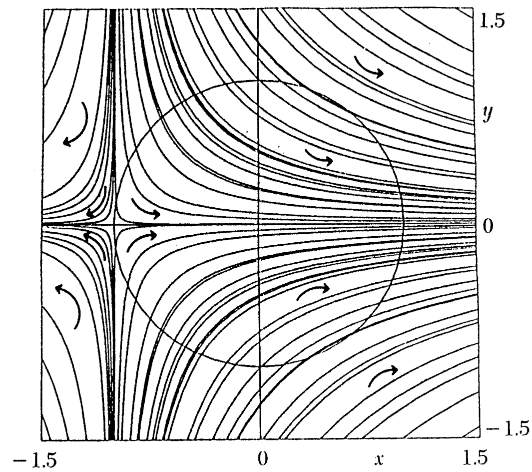


FIGURE 2b
The flow of $\dot{z} = 1/(1+z)$. This function has a simple pole at $z = -1$. This becomes a saddle in flow diagram.

As part of the calculations used to generate Figure 2b, the pole of $F(z)$ becomes a critical point of the corresponding differential equations. Because $z = -1$ is a simple pole, this critical point must be a saddle as is seen in Figure 2b. This idea will be made more precise in a subsequent section.

Figure 2c shows the flow associated with the 11th partial sum of the Maclaurin expansion of $F(z) = 1/(1+z)$, namely

$$\dot{z} = \sum_{n=0}^{11} (-1)^n z^n.$$

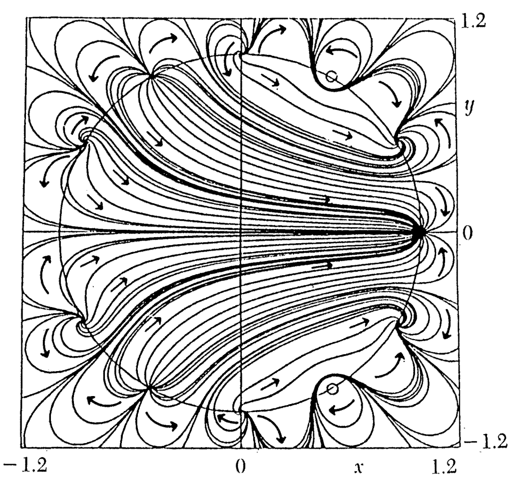


FIGURE 2c
The flow of the 11th partial sum of the Maclaurin expansion of $1/(1+z)$. The 11 zeros of this polynomial all lie on the unit circle and are sinks, sources, or centers.

One can show using the geometric series that all 11 zeros of this 11th-degree polynomial lie on the unit circle. The unit circle also defines the region of convergence for the associated power series. One might speculate that this relationship is indicative of the boundary of the disk of convergence. However,

$$\frac{d}{dz}\left(\frac{1}{1+z}\right)=\sum_{n=1}^{\infty} n(-1)^{n} z^{n-1}$$

also converges on the open unit disk, yet the zeros of each partial sum of this series all lie in the interior of the unit disk (this follows from Rouché's theorem). FIGURE 2d illustrates the flow of the 11th partial sum of this series.

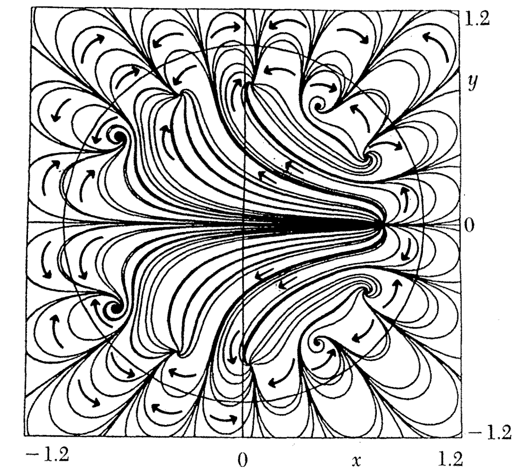


FIGURE 2d

The flow of the 11th partial sum of the Maclaurin expansion of $-1/(1+z)^2$. The 11 zeros of this polynomial all lie in the interior of the unit circle and are sinks or sources.

In FIGURE 2c we see a drastic change in the dynamics as one passes through the unit circle. Inside the unit circle the flow is qualitatively similar to the flow illustrated in FIGURE 2b. Exterior to this disk the flow is reminiscent of the flow of e^z (FIGURE 1a) and clearly there is no hope of extending the approximation to this region. This sharp change is not as evident in FIGURE 2d and without sound mathematical reasoning one might guess that this series has a radius of convergence slightly less than one. However, since the series defining the vector field of FIGURE 2d is the derivative of that of FIGURE 2c we know that this is not the case and in fact both series have the same radius of convergence.

Further analysis of the phase plots In this section we discuss in more detail the calculations used in generating the phase portraits illustrated here. Consider the system of differential equations (1′) and suppose that $z_0 = x_0 + iy_0$ is a zero of an analytic function $F(z)$. Then the point $(x, y) = (x_0, y_0)$ is said to be a *critical point* of (1). To determine the nature of this critical point we compute the Jacobian matrix J of the vector field and evaluate at $(x, y) = (x_0, y_0)$. By the Cauchy-Riemann equations,

$$J=\begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix}$$

and if λ_1 and λ_2 are eigenvalues of J, then $\lambda_1 \lambda_2 = \det J = u_x^2 + u_y^2 \geq 0$ with equality if, and only if, $F'(z) = 0$. Thus if $F'(z_0) \neq 0$ the critical points of the vector field are either sinks or sources. These types of critical points occur in FIGURES 2a and c.

Now suppose $F(z) = P(z)/Q(z)$ with P and Q analytic, $P(z_0) \neq 0$ and $Q(z_0) = 0$. Multiply the numerator and denominator of $F(z)$ by $\overline{Q(z)}$ and the result by the scalar $|Q(z)|^2$ to give the geometrically equivalent flow $G(z) = P(z)\overline{Q(z)}$. It follows that the zeros of $\overline{Q(z)}$ are at the poles of F, yet the flow lines of $G(z)$ coincide with those of $F(z)$. Thus the flow of

$$\begin{aligned} \dot{x} &= \mathrm{Re}(G(x+iy)) \\ \dot{y} &= \mathrm{Im}(G(x+iy)) \end{aligned} \tag{5}$$

describes the flow of F where the poles of F are zeros of G. Although the vector field in (5) is no longer analytic, the zeros of G corresponding to zeros of order one of F (i.e. where $P(z) = 0$ and $P'(z) \neq 0$) are sinks and sources, and the zeros of G that are poles of order one of F (i.e. where $Q(z) = 0$ and $Q'(z) \neq 0$) are all saddles in the phase portrait. In FIGURE 2b there is a saddle at $z = -1$ indicative of a pole at $z = -1$ for $F(z) = 1/(1+z)$.

Applying the ideas of index theory to complex flows provides us with a visual aid in determining the multiplicity of a zero or the order of a pole of an analytic function. To compute the index of a zero z_0 of a differential equation we simply walk once counterclockwise around a small circle surrounding z_0, counting the number of times the vector field rotates during our walk. For each counterclockwise rotation of the vector field, we add one and for each clockwise rotation we subtract one. This can be done quite easily if we rotate a pencil so that the pencil is tangent to solution curves as we traverse the circle about z_0.

Let's first consider poles. The method presented in this paper converts poles of F to zeros of the vector field G. If F has a pole of order n at z_0 then G has a zero of order n at z_0 and it follows from the Principle of the Argument [7, p. 420] that the index of G about z_0 is $-n$. FIGURE 3 illustrates the flow of $\dot{z} = G(z)$ obtained from

$$F(z) = \frac{(z-1)(z+1)^2}{z-i},$$

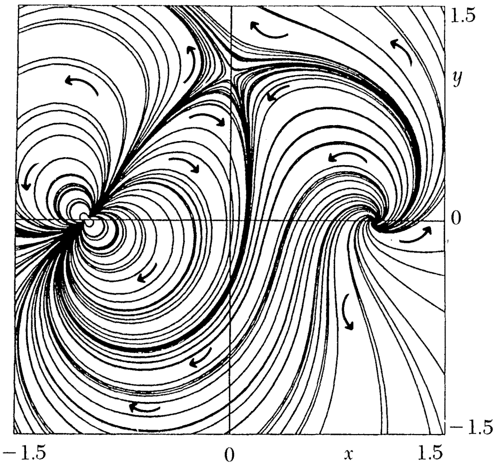


FIGURE 3
The flow of $\dot{z} = (z-1)(z+1)^2/(z-i)$.

having a pole of order one at $z = i$. Note that there are 2, and only 2, invariant curves limiting on the critical point corresponding to the pole of F and trajectories not on these curves are not asymptotic to the critical point. Index theory guarantees that a flow diagram of a function having a pole of order n at z_0 must have exactly $n+1$ invariant curves limiting on z_0. Thus the order of a pole can be determined simply by counting the number of invariant curves.

The same techniques can be applied to zeros of F. If F has a zero of order n at z_0 then the index of the vector field G at z_0 is n. This implies zeros of order one appear as sources, sinks, or centers on our phase portrait. The function illustrated in FIGURE 3 has a zero of order 1 at $z = 1$ and a zero of order 2 at $z = -1$. Hence the index about these critical points is 1 and 2, respectively. What distinguishes these two zeros is the presence of 2 invariant petals about $z = -1$ and none about $z = 1$. In each petal every orbit is heteroclinic to the critical point. This behavior is again guaranteed by index theory. In fact, F has a zero of order n at z_0 if, and only if, there exists $2(n-1)$ invariant petals about z_0. Thus the order of a zero can be determined simply by counting the number of petals about the given zero.

Software considerations The plots in this paper were generated on an IBM-compatible P.C. using a program written in Turbo Basic. This program utilized the Runge-Kutta-Fehlberg RKF4(5) integration method [3, pp. 46–48]. However, there are several other available ODE integration/graphics packages that would give similar results. These plots can be considerably enhanced using the color option where the color is determined at each point plotted either by the length of the velocity vector there, or by whether it was generated for increasing time or decreasing time. The latter procedure aids in determining direction of flow and whether rest points are sinks or sources.

REFERENCES

1. W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 4th edition, John Wiley & Sons, New York, 1986.
2. B. Braden, Picturing functions of a complex variable, College Math J. **16** (1985) 63–72.
3. J. M. A. Danby, *Computing Applications to Differential Equations*, Reston, Reston, VA, 1985.
4. A. D. Gluchoff, Complex power-series–a vector field visualization, this MAGAZINE **66** (1993), 189–191.
5. J. Hale and H. Koçak, *Dynamics and Bifurications*, Springer-Verlag, New York, 1991.
6. K. Hockett and S. Ramamurti, *Dynamics Near the Essential Singularity of a Class of Entire Vector Fields*, Preprint, George Washington University.
7. J. E. Marsden and M. J. Hoffman, *Basic Complex Analysis*, W. H. Freeman, New York, 1987.
8. G. Pólya and G. Latta, *Complex Variables*, John Wiley & Sons, New York, 1974.

On the Asymptotic Stability of a Class of Linear Difference Equations

VASSILIS G. PAPANICOLAOU
Wichita State University
Wichita, KS 67260-0033

1. Introduction The mathematical modeling of a real-life situation often involves difference equations. In population biology in particular, it is quite common that the mathematical model of a phenomenon turns out to be such an equation. C. W. Clark's having a pole of order one at $z = i$. Note that there are 2, and only 2, invariant curves limiting on the critical point corresponding to the pole of F and trajectories not on these curves are not asymptotic to the critical point. Index theory guarantees that a flow diagram of a function having a pole of order n at z_0 must have exactly $n+1$ invariant curves limiting on z_0. Thus the order of a pole can be determined simply by counting the number of invariant curves.

The same techniques can be applied to zeros of F. If F has a zero of order n at z_0 then the index of the vector field G at z_0 is n. This implies zeros of order one appear as sources, sinks, or centers on our phase portrait. The function illustrated in FIGURE 3 has a zero of order 1 at $z = 1$ and a zero of order 2 at $z = -1$. Hence the index about these critical points is 1 and 2, respectively. What distinguishes these two zeros is the presence of 2 invariant petals about $z = -1$ and none about $z = 1$. In each petal every orbit is heteroclinic to the critical point. This behavior is again guaranteed by index theory. In fact, F has a zero of order n at z_0 if, and only if, there exists $2(n-1)$ invariant petals about z_0. Thus the order of a zero can be determined simply by counting the number of petals about the given zero.

Software considerations The plots in this paper were generated on an IBM-compatible P.C. using a program written in Turbo Basic. This program utilized the Runge-Kutta-Fehlberg RKF4(5) integration method [3, pp. 46–48]. However, there are several other available ODE integration/graphics packages that would give similar results. These plots can be considerably enhanced using the color option where the color is determined at each point plotted either by the length of the velocity vector there, or by whether it was generated for increasing time or decreasing time. The latter procedure aids in determining direction of flow and whether rest points are sinks or sources.

REFERENCES

1. W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 4th edition, John Wiley & Sons, New York, 1986.
2. B. Braden, Picturing functions of a complex variable, College Math J. **16** (1985) 63–72.
3. J. M. A. Danby, *Computing Applications to Differential Equations*, Reston, Reston, VA, 1985.
4. A. D. Gluchoff, Complex power-series–a vector field visualization, this MAGAZINE **66** (1993), 189–191.
5. J. Hale and H. Koçak, *Dynamics and Bifurications*, Springer-Verlag, New York, 1991.
6. K. Hockett and S. Ramamurti, *Dynamics Near the Essential Singularity of a Class of Entire Vector Fields*, Preprint, George Washington University.
7. J. E. Marsden and M. J. Hoffman, *Basic Complex Analysis*, W. H. Freeman, New York, 1987.
8. G. Pólya and G. Latta, *Complex Variables*, John Wiley & Sons, New York, 1974.

On the Asymptotic Stability of a Class of Linear Difference Equations

VASSILIS G. PAPANICOLAOU
Wichita State University
Wichita, KS 67260-0033

1. Introduction The mathematical modeling of a real-life situation often involves difference equations. In population biology in particular, it is quite common that the mathematical model of a phenomenon turns out to be such an equation. C. W. Clark's model for the population dynamics of the baleen whale is described by the nonlinear difference equation (see [2])

$$X_{k+1} = \alpha X_k + F(X_{k-n}),$$

where X_k is the number of animals in the k-th period (say year), α is a survival coefficient (typical values are $0, 0.2, 0.4, 0.6, 0.8, 1$—in fact anything between 0 and 1) and $F(X_{k-n})$ represents recruitment, which occurs with a delay of n periods (typical values for n are $0, 1, 2, 3, 4, 5$).

An essential step in the analysis of such equations is the linearization about the equilibrium points (see [3]). If the associated linear equation is *asymptotically stable*, then the corresponding equilibrium point (of the nonlinear equation) is locally absorbing (see [3]). Absence of asymptotic stability implies that it is the nonlinear character that fully controls the behavior of the model. To linearize the equation we set $x_k = X_k - X^*$, where X^* is an equilibrium solution of the nonlinear equation, and then we keep only the terms that are linear in x_k. In Clark's case each equilibrium satisfies

$$X^* = \alpha X^* + F(X^*)$$

and, if we set $\beta = -F'(X^*)$, the linearized equation (near X^*) is

$$x_{k+1} - \alpha x_k + \beta x_{k-n} = 0, \quad \text{where } \alpha, \beta \in R \text{ and } n \geq 1 \text{ is fixed.}$$

The same equation shows up in the linearization of the population model of S. A. Levin and R. M. May (see [6]). Thus, the asymptotic stability of this equation plays an important role in these and possibly in other models.

In this work we obtain a simple necessary and sufficient condition on α and β for the asymptotic stability of the above linear equation. By "asymptotic stability" we mean existence of a unique limit $\lim_k x_k$, independent of initial conditions. Since this is a linear homogeneous equation with constant coefficients, all its solutions are linear combinations of quantities of the form $k^{r-1}\rho^k$, where ρ is a root of the associated characteristic equation

$$z^{n+1} - \alpha z^n + \beta = 0$$

and $r = 1, 2, \ldots, m(\rho)$, $m(\rho)$ being the multiplicity of ρ. It follows that we have asymptotic stability if, and only if, the roots of this algebraic equation have absolute value strictly less than 1 (which, of course, yields $\lim_k x_k = 0$. For more details see [5]).

The case $\alpha = 1$ was solved by Levin and May in [6]; they found a necessary and sufficient condition on β for asymptotic stability. At approximately the same time (1976), Clark [2] obtained a sufficient condition, namely $|\alpha| + |\beta| < 1$, which follows immediately from Rouché's theorem. Recently Kuruklis [4] gave a complete answer to the general case, but his argument is quite complicated. Our approach is simpler, clear and it can be applied to other cases, namely to the equation

$$x_{k+1} - \alpha x_{k-m} + \beta x_{k-n} = 0.$$

2. The necessary and sufficient condition for asymptotic stability As we saw in the introduction, our asymptotic stability question is equivalent to the following

Problem. Let $D = \{z \in C : |z| < 1\}$ and $f(z) = z^{n+1} - \alpha z^n + \beta$, where α and β are real numbers (and $n \geq 1$ is an integer). Find the set A in the $\beta\alpha$-plane such that all the roots of f are in D if and only if (β, α) is in A. In other words, find a (simple) necessary and sufficient condition on the coefficients α and β, in order for the roots of f to have absolute value less than 1.

Observation. The set A has a symmetry. To see that, we set $g(z)=f(-z)$, so that the roots of f are in D if and only if the roots of g are. Then we have the following cases:

(a) If n is even, then $g(z)= -[z^{n+1} - (-\alpha)z^n - \beta]$. Hence

$$(\beta, \alpha) \in A \quad \text{if and only if} \quad (-\beta, -\alpha) \in A,$$

which means that, in this case, A is symmetric with respect to the origin of the $\beta\alpha$-plane.

(b) If n is odd, then $g(z)=z^{n+1} - (-\alpha)z^n + \beta$. Hence

$$(\beta, \alpha) \in A \quad \text{if and only if} \quad (\beta, -\alpha) \in A,$$

which means that, in this case, A is symmetric with respect to the β-axis of the $\beta\alpha$-plane.

Because of the above observation, to solve our problem it is enough to determine the set $A^+=\{(\beta, \alpha) \in A: \alpha \geq 0\}$, i. e. the intersection of A with the upper half (plane) of the $\beta\alpha$-plane. Therefore, the following theorem gives a complete answer to our problem.

THEOREM. *The set A^+ is a "triangle" with two of its sides (straight) line segments, while the third side is curved. Its vertices are $L(1/n, 1+1/n)$, $M(-1,0)$ and $N(1,0)$. The side LM is straight (with equation $\alpha=\beta+1, -1<\beta<1/n$). The side MN is also straight (thus it is the interval $[-1,1]$ of the β-axis), and the "side" NL is the curve with parametric equations (see* FIGURE 1 *below)*

$$\beta=\frac{\sin\theta}{\sin(n\theta)}, \qquad \alpha=\frac{\sin[(n+1)\theta]}{\sin(n\theta)}, \quad \text{where} \quad 0<\theta<\frac{\pi}{n+1}.$$

(Thus NL is straight only if $n=1$. Strictly speaking, the sides LM and NL are pieces of the boundary ∂A^+ of A^+, but are not included in A^+ itself, while the interval $(-1,1)$ of the β-axis, namely the side MN without its endpoints, is in A^+).

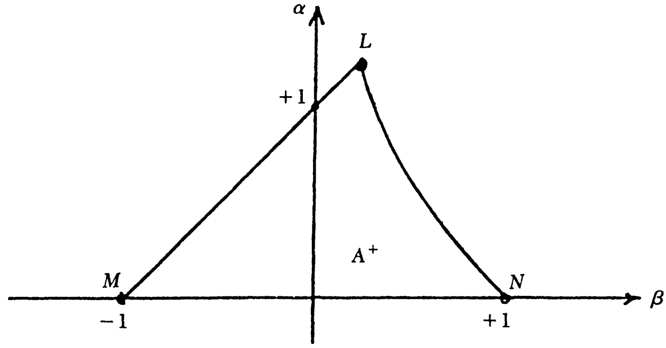


FIGURE 1

Remarks. It is easy to check (see Appendix B) that, if a point is moving on the curve NL, β (and θ) decreases as α increases (unless $n=1$ where β stays constant). The point L corresponds to the value $\theta=0$, while at N we have $\theta=\pi/(n+1)$. The concavity of NL can be checked by a routine calculation. NL is an algebraic curve since $\sin(n\theta)$ is an algebraic function of $\sin\theta$.

Notice that A^+ depends on n. In fact A^+ decreases as n grows. This follows from the fact (which we will see in the proof) that, as n becomes larger, the "side" NL moves to the left. If we let $n \to \infty$ we obtain the limiting set A^+_∞ which is the (honest) triangle $L_\infty MN$, where $L_\infty = (0, 1)$. In particular, the (line) segment $L_\infty N$ lies in A^+ for all n (this is Clark's sufficient condition; see **[2]**).

Finally, as a corollary of the theorem observe that, if we restrict ourselves to the case $\alpha = 1$, we obtain the range of β from the intersection of the line $\alpha = 1$ with A^+. This intersection is a line segment, say PQ, where P lies on LM and Q on NL. The theorem implies that P has coordinates $(0, 1)$. To find the β-coordinate of Q, we set $\alpha = 1$ in the parametric equations of NL. This gives

$$\sin[(n+1)\theta] = \sin(n\theta), \quad \text{hence } (n+1)\theta = \pi - n\theta,$$

since $0 < \theta < \pi/(n+1)$. Therefore $\theta = \pi/(2n+1)$ and

$$\beta = \frac{\sin[\pi/(2n+1)]}{\sin[n\pi/(2n+1)]} = \frac{\sin[2n\pi/(2n+1)]}{\sin[n\pi/(2n+1)]}$$
$$= \frac{2\sin[n\pi/(2n+1)]\cos[n\pi/(2n+1)]}{\sin[n\pi/(2n+1)]} = 2\cos[n\pi/(2n+1)].$$

Thus, $0 < \beta < 2\cos[n\pi/(2n+1)]$, which agrees with the result of Levin and May **[6]**.

Proof of the Theorem. The proof is divided into three steps.

Step 1. Initial bounds for A^+.

The product of the roots of f is $\pm\beta$. We want the roots to be in D, so we must have $|\beta| < 1$. Furthermore, since $f'(z) = (n+1)z^n - n\alpha z^{n-1}$, its roots are 0 (with multiplicity $n-1$) and $n\alpha/(n+1)$. (In fact, the only way that f can have a double root on the unit circle, with $\alpha \geq 0$, is by putting $\alpha = 1 + 1/n$ and $\beta = 1/n$.) If the roots of f are in D, the roots of f' must also be in D, by a nice and simple (but relatively unknown) theorem of Lucas (see **[1]**, p. 29). This implies that

$$|\alpha| < \frac{n+1}{n}.$$

Combining these observations we obtain

$$A^+ \subset (-1, 1) \times [0, 1 + 1/n).$$

From now on, (β, α) will always be assumed to be in $[-1, 1] \times [0, 1 + 1/n]$.

Since we are trying to determine the boundary of A^+, in many cases below we will be interested in the (borderline) situation where one or more roots of f are on the unit circle ∂D and the rest inside. First, we want to examine whether one of these roots is -1. If this is the case then

$$-(-1)^n - \alpha(-1)^n = -\beta, \quad \text{which implies } |\beta| = |1 + \alpha| = 1 + \alpha.$$

This can never happen unless $\alpha = 0$, which is a trivial case.

Step 2. The left boundary of A^+.

Consider the part of the line $\beta = \alpha - 1$ that lies in $[-1, 1] \times [0, 1 + 1/n]$. On this line, f becomes

$$f(z) = z^{n+1} - \alpha z^n + \alpha - 1 = z^n(z-1) - (\alpha - 1)(z^n - 1)$$
$$= (z-1)[z^n - (\alpha - 1)(z^{n-1} + z^{n-2} + \cdots + 1)].$$

Hence, $z_0 = 1$ is a root of f. We want to show that all the other roots are in D. First we treat the case $|\alpha - 1| < 1/n$. If $|z| \geq 1$, then

$$\begin{aligned} &|z^n - (\alpha-1)(z^{n-1} + z^{n-2} + \cdots + 1)| \\ &\quad \geq |z|^n - |\alpha - 1(|z|^{n-1} + |z|^{n-2} + \cdots + 1) \\ &\quad > |z|^n - \frac{1}{n}(|z|^{n-1} + |z|^{n-2} + \cdots + 1) \\ &\quad = \frac{|z|^n - |z|^{n-1}}{n} + \frac{|z|^n - |z|^{n-2}}{n} + \cdots + \frac{|z|^n - 1}{n} > 0. \end{aligned}$$

The remaining case is $0 < \alpha \leq 1 - (1/n)$ (remember that $\alpha = 0$ is a trivial case). To reach this case we start with an α as in the previous case (namely $|\alpha - 1| < 1/n$) and we move it toward 0. As we do that, what can happen? Either

(i) all roots apart from $z_0 = 1$ stay in D; or

(ii) at least one pair of complex conjugate roots z_j and $\bar{z}_j$ reach ∂D (only the root z_0 can become ± 1 because of the discussion in the previous step).

We now show that (ii) cannot happen. Remember that $\beta = \alpha - 1$. Let $z_j = e^{i\theta}$. By taking conjugates, we can assume without loss of generality that $0 < \theta < \pi$. Now

$$e^{i(n+1)\theta} - \alpha e^{in\theta} + \alpha - 1 = 0, \quad \text{which implies } |\alpha - e^{i\theta}| = |\alpha - 1|.$$

But this equality is impossible if α and θ satisfy our conditions. To summarize:

If $0 < \alpha < 1 + 1/n$ and $\beta = \alpha - 1$, then f has one root $z_0 = 1$ and all the rest in D.

Now, fix $\alpha \in (0, 1 + 1/n)$ and move β in the interval $(\alpha - 1 - \varepsilon, \alpha - 1 + \varepsilon)$, where $\varepsilon > 0$ is sufficiently small. Let $z_0(\beta)$ be the root of f that satisfies $z_0(\beta) = 1$ when $\beta = \alpha - 1$. It is well defined as long as ε stays small, by the above discussion and, in fact, stays real because there is no other root very near 1 to play the role of its conjugate. We have

$$z_0(\beta)^{n+1} - \alpha z_0(\beta)^n + \beta = 0.$$

Differentiation with respect to β yields

$$(n+1) z_0^n z_0' - n\alpha z_0^{n-1} z_0' + 1 = 0.$$

At $\beta = \alpha - 1$, $z_0 = 1$, therefore

$$(n+1) z_0'(\alpha - 1) - n\alpha z_0'(\alpha - 1) + 1 = 0 \quad \text{or} \quad z_0'(\alpha - 1)(n + 1 - n\alpha) = -1.$$

Since $\alpha \in (0, 1 + 1/n)$ we can conclude that $z_0'(\alpha - 1) < 0$. This implies that if $\beta \in (\alpha - 1 - \varepsilon, \alpha - 1)$, then $|z_0(\beta)| > 1$, whereas if $\beta \in (\alpha - 1, \alpha - 1 + \varepsilon)$, then $z_0(\beta) \in D$. Thus, in order to establish that the left boundary of A^+ is the segment $\beta = \alpha - 1$ with $0 < \alpha \leq 1 + 1/n$ we still need to show that there is no β in $(-1, \alpha - 1)$ for which f has all its roots in D. To see that, let β move from $\alpha - 1$ toward -1. When it is near $\alpha - 1$, z_0 is outside D and all the other roots of f are in D. As β approaches -1, other roots might pass ∂D. We have seen that as they pass, they cannot be real, thus they pass in complex conjugate pairs. Conclusion: If β is in $(-1, \alpha - 1)$ there is always an odd number of roots outside D.

Step 3. The right boundary of A^+.

I. A similar approach is used for determining the right boundary of A^+, but the situation here is a little more complicated since, as β moves to the right, the first root that leaves D cannot be real (as we have already seen).

Again we fix α in $(0, 1 + 1/n)$ and then we move β to the right, starting from $\beta = \alpha - 1$. At certain values of $\beta \le 1$, two roots z_j and $\bar{z}_j$ of hit ∂D. We set $z_j = e^{i\theta}$, where $0 < \theta < \pi$ (without loss of generality). Hence

$$e^{i(n+1)\theta} - \alpha e^{in\theta} + \beta = 0,$$

which implies that

$$\beta = -e^{in\theta}(e^{i\theta} - \alpha), \tag{1}$$

and

$$|\beta| = |\alpha - e^{i\theta}| > |\alpha - 1|.$$

Since $\beta > \alpha - 1$ we must have

$$|\alpha - 1| < \beta \le 1$$

and therefore

$$\beta = |\alpha - e^{i\theta}|. \tag{2}$$

In particular, the upper bound of β restricts θ to the interval $(0, \pi/2)$. Taking arguments in (1) we get (since $\beta > 0$)

$$\arg(\beta) = 2k\pi = \pi + n\theta + \phi, \quad \text{where } \phi = \arg(e^{i\theta} - \alpha). \tag{3}$$

Without loss of generality we can always assume that

$$0 < \phi < \pi.$$

Notice that (1) is equivalent to (2) and (3) together. On the other hand, (1) is the necessary and sufficient condition for a root of f to be on ∂D. The bounds for θ and ϕ imply that the integer k that appears in (3) is between 1 and $1 + n/4$.

The geometric picture is helpful (see FIGURE 2): Consider the triangle in the complex plane whose vertices are 0, α and $z_j = e^{i\theta}$. The angle corresponding to the vertex 0 is θ, while the exterior angle at the vertex α is ϕ (or, the angle corresponding to α is $\pi - \phi$). In particular $\theta < \phi$.

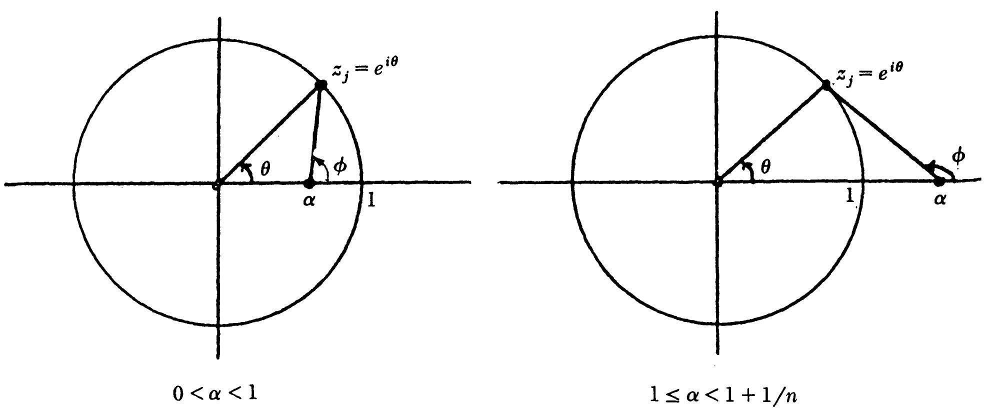


FIGURE 2

It follows that, given $\alpha \in (0, 1 + 1/n)$, to determine a β in $(|\alpha - 1|, 1]$ for which f has a root on ∂D, we first fix an integer in the range given above, then we find the θ in $(0, \pi/2)$ that satisfies (3) and, finally, we obtain our β from (2). From (3) it follows easily that bigger k's give bigger θ's, which, then, give bigger β's, as can be seen from (2). Therefore, the smallest β in $(|\alpha - 1|, 1]$ for which f has a root on ∂D must be obtained by setting $k = 1$ in (3), which then becomes

$$n\theta + \phi = \pi, \quad \text{where } \phi = \arg(e^{i\theta} - \alpha). \tag{3'}$$

Let us call $\beta_r(\alpha)$ this particular β (r here stands for "right", since this will turn out to be right boundary of A^+, at a fixed α). If $\alpha \in (0, 1)$ such β_r obviously exists (uniquely) because, if θ is close to 0, then ϕ is close to 0 too and, by increasing θ we will be able to satisfy (3′) and then get our β_r from (2). But if $\alpha \in [1, 1 + 1/n)$ things get a little trickier because, as $\theta \downarrow 0$, then $\phi \uparrow \pi$ (unless $\alpha = 1$, where ϕ does not have a limit; see FIGURE 2) and, thus, (3′) may not have any solution θ in $(0, \pi/2)$. To see that this is not the case, observe that, if $\theta > 0$ is very close to 0, then

$$\pi - \phi = \frac{\theta}{\alpha - 1} + O(\theta^2) = \frac{\theta}{1/n} + \delta\theta + O(\theta^2) = n\theta + \delta\theta + O(\theta^2),$$

where $\delta = (\alpha - 1)^{-1} - n > 0$. Hence, the above formula implies that

$$n\theta + \phi = \pi - \delta\theta + O(\theta^2),$$

which says that, if θ is sufficiently small (but strictly positive), then $n\theta + \phi < \pi$. By increasing θ we can then satisfy (3′) and obtain our β_r. For such a θ, because of (3′), we must have

$$0 < \theta < \frac{\pi}{n+1}.$$

The law of the sines in the triangle with vertices 0, α and $z_j = e^{i\theta}$ yields

$$\frac{\beta_r}{\sin\theta} = \frac{1}{\sin\phi} = \frac{\alpha}{\sin(\phi - \theta)}.$$

But, by (3′), $\sin\phi = \sin(n\theta)$ and $\sin(\phi - \theta) = \sin[(n+1)\theta]$. Hence

$$\beta_r = \frac{\sin\theta}{\sin(n\theta)}, \tag{4}$$

where θ can be computed from the formulas

$$\alpha = \frac{\sin[(n+1)\theta]}{\sin(n\theta)}, \qquad 0 < \theta < \frac{\pi}{n+1}. \tag{5}$$

Since for $n \geq 2$, after a little algebra, $d\beta_r/d\alpha$ can be shown to be strictly negative (see Appendix B) for this range of θ, we have a unique β_r, for any α in $(0, 1 + 1/n)$. Finally, as n gets bigger, the solution θ of (3′) obviously gets smaller and, thus, β_r, obtained from (2), gets smaller too (i.e., the right boundary of A^+ moves to the left). Since $\alpha \in (0, 1 + 1/n)$ and $|\alpha - 1| < \beta_r \leq 1$, (3′) implies that, in the limit $n \to \infty$, $\beta_r(\alpha) = 1 - \alpha$, where $0 < \alpha \leq 1$.

In the other extreme case, namely $n = 1$, our formulas imply that $\beta_r(\alpha) = 1$, where $0 < \alpha \leq 2$.

II. In order to establish that $\beta_r(\alpha)$ is actually the right boundary of A^+, we must show that, if β satisfies

$$\beta_r(\alpha) < \beta \leq 1, \tag{6}$$

then there are roots of f lying outside D. To prove that, we just need to show that, if for some β, one root z_j is on the unit circle ∂D, then, by increasing β, this root moves away from D. To make it more precise, let $z_j = re^{i\theta}$ be a root of f, which, of course, depends on β. We want to show that, if at some β we have that $r(\beta) = |z_j| = 1$, then $r'(\beta) > 0$, for this particular β, which, of course, must satisfy (6).

Since z_j is a root of f,

$$r^{n+1}e^{i(n+1)\theta} - \alpha r^n e^{in\theta} + \beta = 0.$$

Differentiating implicitly with respect to β and then setting $r = 1$ we obtain

$$(n+1)r'e^{i(n+1)\theta} + e^{i(n+1)\theta}i(n+1)\theta' - \alpha n r' e^{in\theta} - \alpha e^{in\theta} in\theta' + 1 = 0.$$

If we set $r = 1$ in the first equation and then write the corresponding equations for the real and imaginary parts, the above two equations become

$$\cos[(n+1)\theta] - \alpha\cos(n\theta) + \beta = 0, \tag{7}$$

$$\sin[(n+1)\theta] = \alpha\sin(n\theta), \tag{8}$$

$$\begin{aligned}(n+1)r'\cos[(n+1)\theta] &- (n+1)\theta'\sin[(n+1)\theta] \\ &- \alpha n r'\cos(n\theta) + \alpha n\theta'\sin(n\theta) + 1 = 0\end{aligned} \tag{9}$$

and

$$\begin{aligned}(n+1)r'\sin[(n+1)\theta] &+ (n+1)\theta'\cos[(n+1)\theta] \\ &- \alpha n r'\sin(n\theta) - \alpha n\theta'\cos(n\theta) = 0.\end{aligned} \tag{10}$$

The last equation implies

$$\theta'[(n+1)\cos[(n+1)\theta] - \alpha n\cos(n\theta)] = r'[\alpha n\sin(n\theta) - (n+1)\sin[(n+1)\theta]],$$

or

$$\theta' = \frac{\sin[(n+1)\theta]}{n\beta - \cos[(n+1)\theta]}r', \tag{11}$$

thanks to (7) and (8). Likewise, using (7) and (8) in (9) we obtain

$$r'[n\beta - \cos[(n+1)\theta]] + \theta'\sin[(n+1)\theta] = 1,$$

which after applying (11), becomes

$$r'\left[n\beta - \cos[(n+1)\theta] - \frac{\sin^2[(n+1)\theta]}{n\beta - \cos[(n+1)\theta]}\right] = 1. \tag{12}$$

Remember that we are interested in the sign of r'. The above equation tells us that $r' > 0$ if and only if $n\beta - \cos[(n+1)\theta] > 0$. But, by (6) and (4)

$$n\beta > n\beta_r = \frac{n\sin\theta}{\sin(n\theta)} \geq 1 \tag{13}$$

(for the last inequality see Appendix A) and, therefore we have established that $r' < 0$. This finishes the proof of the theorem.

Remark. The polynomial $f(z) = z^{n+1} - \alpha z^{n-k} + \beta$, where $1 \le k \le n-1$, can be treated in a similar way.

3. Concluding Remarks As we saw in the introduction, the asymptotic stability of the equation

$$x_{k+1} - \alpha x_k + \beta x_{k-n} = 0$$

plays an important role in certain population biology models. In this work we showed that this equation is asymptotically stable if, and only if, the point (β, α) lies in the set A of the $\beta\alpha$-plane where

(i) A is symmetric with respect to the origin, if n is even;

(ii) A is symmetric with respect to the β-axis, if n is odd and, in both cases, the upper-half A^+ of A (namely the part that lies in the upper half-plane $\alpha \ge 0$) has a simple explicit description given by the theorem in the beginning of Sec. 2.

Of course, the parameters α, β and n are determined from the corresponding "real world" situation and then our criterion can give an immediate answer to the question of whether the population under study will approach an equilibrium or not. Typical values for α and n are, for example, $\alpha = .8$, $n = 4$ (see **[2]** and our introduction for the meaning of these numbers). Then our theorem, with the help of Newton's method for solving numerically transcendental equations, gives $\theta = 0.4496$. Thus the corresponding equilibrium solution of the nonlinear model is locally absorbing if $-0.2 < \beta < 0.446$.

Appendix A

The last inequality appearing in (13) follows immediately from the following lemma.

LEMMA. *Let* $\theta \in [0, \pi/2]$ *and* $x \ge 1$. *Then*

$$x \sin \theta \ge \sin(x\theta).$$

Proof. Fix $x \ge 1$. Define

$$g(\theta) = x \sin \theta - \sin(x\theta), \qquad \theta \in [0, \pi/2],$$

and notice that $g(0) = 0$ and $g(\theta_m) \ge 0$, where θ_m is any zero of g' in $[0, \pi/2]$.

Appendix B

For α and β_r given parametrically by (4) and (5) we can compute easily that

$$\frac{d\beta_r}{d\alpha} = -\frac{\sin[(n-1)\theta] - (n-1)\sin\theta\cos(n\theta)}{n\sin\theta - \cos[(n+1)\theta]\sin(n\theta)}, \qquad 0 < \theta < \frac{\pi}{n+1},$$

or

$$\frac{d\beta_r}{d\alpha} = -\frac{(n+1)\sin[(n-1)\theta] - (n-1)\sin[(n+1)\theta]}{(2n+1)\sin\theta - \sin[(2n+1)\theta]}. \qquad (B1)$$

In particular

$$\lim_{\theta \downarrow 0} \frac{d\beta_r}{d\alpha} = -\frac{n-1}{2n+1} \quad \text{and} \quad \left.\frac{d\beta_r}{d\alpha}\right|_{\theta=\pi/(n+1)} = -\cos\frac{\pi}{n+1}.$$

The case $n = 1$ is trivial, so let us assume that $n \geq 2$. The denominator in $(B1)$ is nonnegative by the lemma of Appendix A. In fact, it is easy to check that it is strictly positive. Let $N(\theta)$ be the numerator appearing in $(B1)$, namely

$$N(\theta) = (n+1)\sin[(n-1)\theta] - (n-1)\sin[(n+1)\theta].$$

We have that $N(0) = 0$ and

$$N'(\theta) = 2(n^2-1)\sin\theta\sin(n\theta) > 0.$$

Thus $N(\theta) > 0$, for $\theta \in (0, \pi/(n+1))$, and therefore β_r is a (strictly) decreasing function of α.

Acknowledgment. The author wishes to thank professor Gerry Ladas for proposing the problem to him; also, for his encouragement and suggestions. The author also wishes to thank the National Science Foundation for partially supporting him (Grant DMS-9011641). Finally, we want to thank the former editor of this MAGAZINE for her numerous constructive suggestions, which made the revised version considerably more clear and readable.

REFERENCES

1. Ahlfors, Lars V., *Complex Analysis*, 3rd edition, McGraw-Hill Book Co., New York, 1979.
2. Clark C. W., A delay-recruitment model of population dynamics with an application to baleen whale populations, *J. Math. Biol.* **3** (1976), 381–391.
3. Gyori I. and Ladas G., *Oscillation Theory of Delay Difference Equations with Applications*, Clarendon Press (Oxford University Press), Oxford, 1992.
4. Kuruklis Spiridon A., The Asymptotic Stability of $x_{n+1} - \alpha x_n + bx_{n-k} = 0$, preprint.
5. Lakshmikantham V. and Trigiante D., *Theory of Difference Equations: Numerical Methods and Applications*, Academic Press Inc., San Diego, CA, 1988.
6. Levin S. A. and May R. M., A note on difference-delay equations, *Theor. Pop. Bio.* **9** (1976), 178–187.

Math Bite: A Generalization of a Curiosity that Feynman Remembered All His Life

If a boy named Morrie Jacobs told him that the cosine of 20 *degrees multiplied by the cosine of* 40 *degrees multiplied by the cosine of* 80 *degrees equaled exactly one-eighth, he would remember that curiosity for the rest of his life, and he would remember that he was standing in Morrie's father's leather shop when he learned it* **[1]**.

Morrie Jacobs' identity is the special case $k = 3, a = 20^o$, of the following identity that follows by induction on k using $\sin 2b = 2\sin b\cos b$, with $b = 2^{k-1}a$.

$$2^k \prod_{j=0}^{k-1} \cos(2^j a) = \frac{\sin(2^k a)}{\sin a}.$$

For additional material, consult Anderson **[2]**.

In particular

$$\lim_{\theta \downarrow 0} \frac{d\beta_r}{d\alpha} = -\frac{n-1}{2n+1} \quad \text{and} \quad \left.\frac{d\beta_r}{d\alpha}\right|_{\theta=\pi/(n+1)} = -\cos\frac{\pi}{n+1}.$$

The case $n = 1$ is trivial, so let us assume that $n \geq 2$. The denominator in $(B1)$ is nonnegative by the lemma of Appendix A. In fact, it is easy to check that it is strictly positive. Let $N(\theta)$ be the numerator appearing in $(B1)$, namely

$$N(\theta) = (n+1)\sin[(n-1)\theta] - (n-1)\sin[(n+1)\theta].$$

We have that $N(0) = 0$ and

$$N'(\theta) = 2(n^2-1)\sin\theta\sin(n\theta) > 0.$$

Thus $N(\theta) > 0$, for $\theta \in (0, \pi/(n+1))$, and therefore β_r is a (strictly) decreasing function of α.

Acknowledgment. The author wishes to thank professor Gerry Ladas for proposing the problem to him; also, for his encouragement and suggestions. The author also wishes to thank the National Science Foundation for partially supporting him (Grant DMS-9011641). Finally, we want to thank the former editor of this MAGAZINE for her numerous constructive suggestions, which made the revised version considerably more clear and readable.

REFERENCES

1. Ahlfors, Lars V., *Complex Analysis*, 3rd edition, McGraw-Hill Book Co., New York, 1979.
2. Clark C. W., A delay-recruitment model of population dynamics with an application to baleen whale populations, *J. Math. Biol.* **3** (1976), 381–391.
3. Gyori I. and Ladas G., *Oscillation Theory of Delay Difference Equations with Applications*, Clarendon Press (Oxford University Press), Oxford, 1992.
4. Kuruklis Spiridon A., The Asymptotic Stability of $x_{n+1} - \alpha x_n + bx_{n-k} = 0$, preprint.
5. Lakshmikantham V. and Trigiante D., *Theory of Difference Equations: Numerical Methods and Applications*, Academic Press Inc., San Diego, CA, 1988.
6. Levin S. A. and May R. M., A note on difference-delay equations, *Theor. Pop. Bio.* **9** (1976), 178–187.

Math Bite: A Generalization of a Curiosity that Feynman Remembered All His Life

If a boy named Morrie Jacobs told him that the cosine of 20 *degrees multiplied by the cosine of* 40 *degrees multiplied by the cosine of* 80 *degrees equaled exactly one-eighth, he would remember that curiosity for the rest of his life, and he would remember that he was standing in Morrie's father's leather shop when he learned it* **[1]**.

Morrie Jacobs' identity is the special case $k = 3, a = 20^o$, of the following identity that follows by induction on k using $\sin 2b = 2\sin b \cos b$, with $b = 2^{k-1}a$.

$$2^k \prod_{j=0}^{k-1} \cos(2^j a) = \frac{\sin(2^k a)}{\sin a}.$$

For additional material, consult Anderson **[2]**.

REFERENCES

1. James Gleick, *Genius: The Life and Science of Richard Feynman*, Pantheon Books, New York 1992, page 47.
2. E. C. Anderson, *Morrie's law and experimental mathematics*, to appear, J. of Recreational Mathematics.

—W. A. BEYER AND J. D. LOUCK
LOS ALAMOS NATIONAL LABORATORY
LOS ALAMOS, NM 87545
—D. ZEILBERGER
TEMPLE UNIVERSITY
PHILADELPHIA, PA 19122

Building Home Plate: Field of Dreams or Reality?

MICHAEL J. BRADLEY
Merrimack College
North Andover, MA 01845

In the movie *Field of Dreams*, Kevin Costner's character, Ray Kinsella, considers building a baseball park in the middle of his cornfield. "If you build it, they will come," encourages a voice from the past. As an assistant coach for my nine-year-old son's baseball team, I was interested to read in the official league rules the following specifications for home plate:

> "*Home base shall be marked by a five-sided slab of whitened rubber. It shall be a* 12-*inch square with two of the corners filled in so that one edge is* 17 *inches long, two are* 8 1/2 *inches and two are* 12 *inches.*" ([**1**], *p.* 160)

An accompanying diagram shows the finished product:

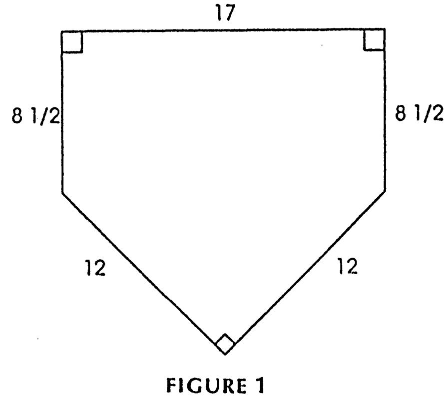


FIGURE 1

Pondering these instructions, I wondered not whether we *should* build it, but whether we *could* build it. Is such a home plate possible?

The "correct" answer is "No." The figure implies the existence of a right isosceles triangle with sides 12, 12 and 17. But (12, 12, 17) is not (quite) a Pythagorean triple: $12^2 + 12^2 = 288$; $17^2 = 289$. Thus, these specifications seem to give new meaning to a "Field of Dreams."

REFERENCES

1. James Gleick, *Genius: The Life and Science of Richard Feynman*, Pantheon Books, New York 1992, page 47.
2. E. C. Anderson, *Morrie's law and experimental mathematics*, to appear, J. of Recreational Mathematics.

—W. A. BEYER AND J. D. LOUCK
LOS ALAMOS NATIONAL LABORATORY
LOS ALAMOS, NM 87545
—D. ZEILBERGER
TEMPLE UNIVERSITY
PHILADELPHIA, PA 19122

Building Home Plate: Field of Dreams or Reality?

MICHAEL J. BRADLEY
Merrimack College
North Andover, MA 01845

In the movie *Field of Dreams*, Kevin Costner's character, Ray Kinsella, considers building a baseball park in the middle of his cornfield. "If you build it, they will come," encourages a voice from the past. As an assistant coach for my nine-year-old son's baseball team, I was interested to read in the official league rules the following specifications for home plate:

> *"Home base shall be marked by a five-sided slab of whitened rubber. It shall be a 12-inch square with two of the corners filled in so that one edge is 17 inches long, two are 8 1/2 inches and two are 12 inches."* ([1], *p.* 160)

An accompanying diagram shows the finished product:

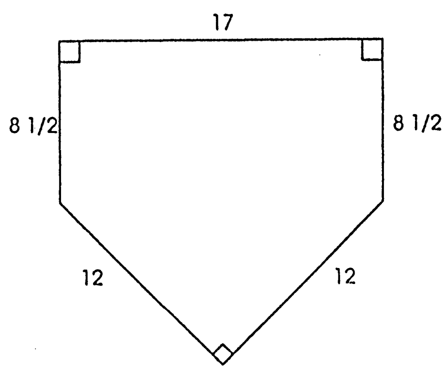


FIGURE 1

Pondering these instructions, I wondered not whether we *should* build it, but whether we *could* build it. Is such a home plate possible?

The "correct" answer is "No." The figure implies the existence of a right isosceles triangle with sides 12, 12 and 17. But (12, 12, 17) is not (quite) a Pythagorean triple: $12^2 + 12^2 = 288$; $17^2 = 289$. Thus, these specifications seem to give new meaning to a "Field of Dreams."

On the other hand, if one interprets the values 12 and 17 as measured numbers, accurate to two significant digits, then home plate *can* be built, since, to that degree of accuracy, (12, 12, 17) *is* a Pythagorean triple.

We *can* build it! Let them come.

REFERENCE

1. Peter Kreutzer and Ted Kerley, *Little League's Official How-to-Play Baseball Book*, Doubleday, New York, 1990.

The Pythagorean Proposition: A Proof by Means of Calculus

MIKE STARING
Hogeschool Katholieke Leergangen
5022 DN Tilburg, The Netherlands

E. S. Loomis, in [1], argues that there can be no trigonometric proof nor any proof based on analytical geometry or calculus for the Pythagorean proposition because each of these subjects "accepts the truth of geometry as established, and therefore furnishes no new proof." His argument seems valid in so far as functions of two (or more) variables are involved in such a 'proof.' Since calculus of one variable can be developed without using the Pythagorean theorem, circular reasoning may be avoided. The following is a proof of the proposition using calculus.

Let ABC be a triangle with its right angle at A. Keep AB fixed and let $AB = b$. Denote AC by the variable x, so that BC is a function of x, $f(x)$. See FIGURE 1. If AC increases by an amount Δx, then BC will increase by Δf. From similar triangles,

$$\frac{\Delta f}{\Delta x} = \frac{CQ}{CD} > \frac{CP}{CD} = \frac{CA}{CB} = \frac{x}{f(x)}.$$

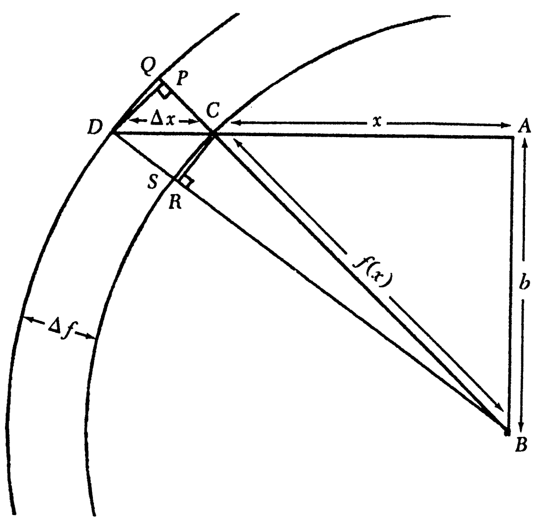


FIGURE 1

On the other hand, if one interprets the values 12 and 17 as measured numbers, accurate to two significant digits, then home plate *can* be built, since, to that degree of accuracy, (12, 12, 17) *is* a Pythagorean triple.

We *can* build it! Let them come.

REFERENCE

1. Peter Kreutzer and Ted Kerley, *Little League's Official How-to-Play Baseball Book*, Doubleday, New York, 1990.

The Pythagorean Proposition: A Proof by Means of Calculus

MIKE STARING
Hogeschool Katholieke Leergangen
5022 DN Tilburg, The Netherlands

E. S. Loomis, in [1], argues that there can be no trigonometric proof nor any proof based on analytical geometry or calculus for the Pythagorean proposition because each of these subjects "accepts the truth of geometry as established, and therefore furnishes no new proof." His argument seems valid in so far as functions of two (or more) variables are involved in such a 'proof.' Since calculus of one variable can be developed without using the Pythagorean theorem, circular reasoning may be avoided. The following is a proof of the proposition using calculus.

Let ABC be a triangle with its right angle at A. Keep AB fixed and let $AB = b$. Denote AC by the variable x, so that BC is a function of x, $f(x)$. See FIGURE 1. If AC increases by an amount Δx, then BC will increase by Δf. From similar triangles,

$$\frac{\Delta f}{\Delta x} = \frac{CQ}{CD} > \frac{CP}{CD} = \frac{CA}{CB} = \frac{x}{f(x)}.$$

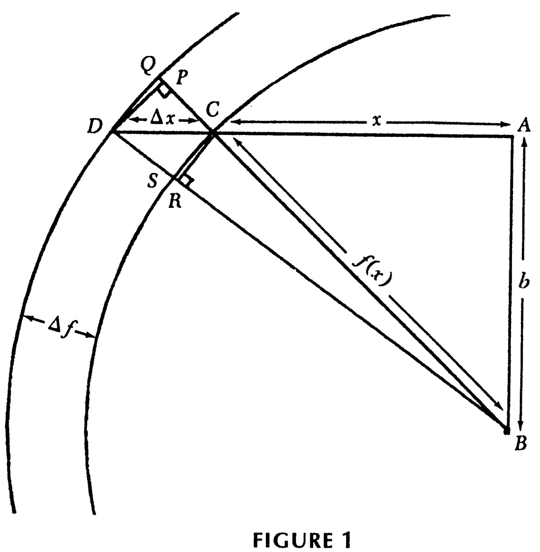


FIGURE 1

But also

$$\frac{\Delta f}{\Delta x}=\frac{SD}{CD}<\frac{RD}{CD}=\frac{AD}{BD}=\frac{x+\Delta x}{f(x)+\Delta f}<\frac{x}{f(x)}+\frac{\Delta x}{f(x)}.$$

Now let $\Delta x \to 0^+$; we find

$$\frac{df}{dx}=\frac{x}{f(x)}.$$

(Although only the case that $\Delta x > 0$ has been studied here, it is easy to derive a similar equation for $\Delta x < 0$.)

Hence, f is a differentiable function of x. Solving the differential equation yields

$$f^2(x)=x^2+c,$$

where c is a constant. Obviously, if $x=0$ then $f(x)=b$. Hence $c=b^2$. The proof of the Pythagorean proposition is complete.

REFERENCE

1. Elisha Scott Loomis, *The Pythagorean Proposition*, National Council of Teachers of Mathematics, Washington, DC, 1968, pp. 244–245.

Update on William Wernick's "Triangle Constructions with Three Located Points"

LEROY F. MEYERS*
The Ohio State University
Columbus, OH 43210

William Wernick's paper [9] contains a list of 139 problems, each of which asks for the Euclidean construction of a triangle from triples of "located points", i.e., points such as vertices, feet of altitudes, centroids, etc., whose location is given. Wernick was able to resolve nearly two-thirds of these problems, either by finding constructions or by proving redundancy of the data. (See figures and complete list of located points below.)

The notation below, introduced in [9], will be used in what follows for various points associated with a triangle. (See FIGURES 1–4.)

A, B, C, O	the three vertices, and circumcenter;
M_a, M_b, M_c, G	feet of the three medians, and centroid;
H_a, H_b, H_c, H	feet of the three altitudes, and orthocenter;
T_a, T_b, T_c, I	feet of the three internal angle bisectors, and incenter.

*We report with regret that Professor Meyers, the author of this article, died suddenly in November, 1995. Professor Meyers had a long-standing interest in geometry, problems, and undergraduate education. In the week he died, Professor Meyers was scheduled to speak—on topics related to this article—to a student mathematics club at Ohio State. During the period 1975–85, he served as Associate Editor of this MAGAZINE, including six years as Associate Problems Editor. —Ed.

But also

$$\frac{\Delta f}{\Delta x}=\frac{SD}{CD}<\frac{RD}{CD}=\frac{AD}{BD}=\frac{x+\Delta x}{f(x)+\Delta f}<\frac{x}{f(x)}+\frac{\Delta x}{f(x)}.$$

Now let $\Delta x \to 0^+$; we find

$$\frac{df}{dx}=\frac{x}{f(x)}.$$

(Although only the case that $\Delta x>0$ has been studied here, it is easy to derive a similar equation for $\Delta x<0$.)

Hence, f is a differentiable function of x. Solving the differential equation yields

$$f^2(x)=x^2+c,$$

where c is a constant. Obviously, if $x=0$ then $f(x)=b$. Hence $c=b^2$. The proof of the Pythagorean proposition is complete.

REFERENCE

1. Elisha Scott Loomis, *The Pythagorean Proposition*, National Council of Teachers of Mathematics, Washington, DC, 1968, pp. 244–245.

Update on William Wernick's "Triangle Constructions with Three Located Points"

LEROY F. MEYERS*
The Ohio State University
Columbus, OH 43210

William Wernick's paper **[9]** contains a list of 139 problems, each of which asks for the Euclidean construction of a triangle from triples of "located points", i.e., points such as vertices, feet of altitudes, centroids, etc., whose location is given. Wernick was able to resolve nearly two-thirds of these problems, either by finding constructions or by proving redundancy of the data. (See figures and complete list of located points below.)

The notation below, introduced in **[9]**, will be used in what follows for various points associated with a triangle. (See FIGURES 1–4.)

A, B, C, O	the three vertices, and circumcenter;
M_a, M_b, M_c, G	feet of the three medians, and centroid;
H_a, H_b, H_c, H	feet of the three altitudes, and orthocenter;
T_a, T_b, T_c, I	feet of the three internal angle bisectors, and incenter.

*We report with regret that Professor Meyers, the author of this article, died suddenly in November, 1995. Professor Meyers had a long-standing interest in geometry, problems, and undergraduate education. In the week he died, Professor Meyers was scheduled to speak—on topics related to this article—to a student mathematics club at Ohio State. During the period 1975–85, he served as Associate Editor of this MAGAZINE, including six years as Associate Problems Editor. —Ed.

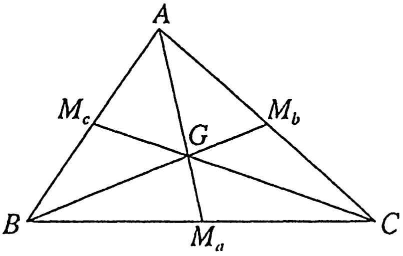


FIGURE 1

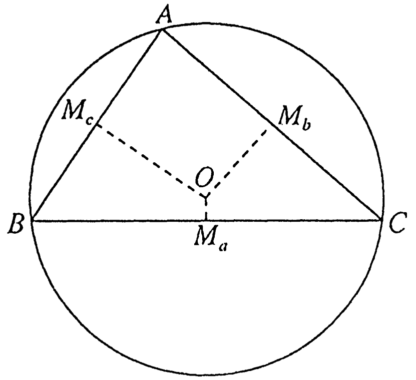


FIGURE 2

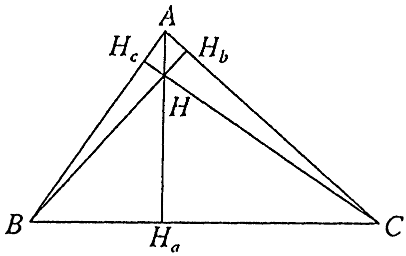


FIGURE 3

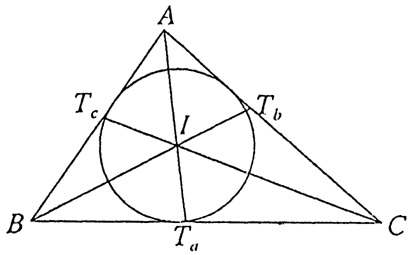


FIGURE 4

Since the appearance of Wernick's paper more than 13 years ago, about half of the problems left unresolved by him have been resolved, some positively but most negatively, and this paper is a report on the new results.

Table 1 contains a listing of these recently resolved problems, numbered as in **[9]**, together with their resolutions. (A misprint in problem 102 is corrected.) Twenty problems remain unresolved.

TABLE 1 For each of the 30 triples of points listed, the problem of constructing the corresponding triangle ABC has been resolved by the author since the appearance of **[9]**. The triples are numbered as in that article. The letters **S**, **U**, and **L** designate that the problem of constructing a triangle from the given triple by Euclidean means is **S**olvable, **U**nsolvable, or **L**ocus-restricted, the last meaning that for a triangle to exist, one of the points must lie on a locus curve determined by the other two, but is not determined completely.

26.	A, M_a, T_b	**U**	58.	A, T_a, T_b	**S**	80.	O, H, I	**U**	96.	M_a, G, I	**S**	114.	M_a, T_b, I	**U**
27.	A, M_a, I	**S**	68.	O, M_a, T_b	**U**	82.	O, T_a, I	**S**	100.	M_a, H_a, T_b	**U**	115.	G, H_a, H_b	**U**
42.	A, G, T_b	**U**	72.	O, G, T_a	**U**	87.	M_a, M_b, H	**S**	102.	M_a, H_b, H_c	**L**	120.	G, H, T_a	**U**
43.	A, G, I	**S**	73.	O, G, I	**U**	88.	M_a, M_b, T_a	**U**	106.	M_a, H_b, T_c	**U**	121.	G, H, I	**U**
56.	A, H, T_b	**U**	74.	O, H_a, H_b	**U**	89.	M_a, M_b, T_c	**U**	107.	M_a, H_b, I	**U**	130.	H_a, H, T_b	**U**
57.	A, H, I	**S**	79.	O, H, T_a	**U**	95.	M_a, G, T_b	**U**	108.	M_a, H, T_a	**U**	131.	H_a, H, I	**S**

It is an interesting challenge to verify the results shown in the table. One sample verification is given below; the remaining verifications (and extensions!) are left to the interested reader, who may obtain further information from the author.

Algebra, often in connection with analytic geometry, can be used to prove that there is no Euclidean construction from certain triples of located points. All such proofs proceed by contradiction, and depend on Gauss's criterion for Euclidean constructibility. The following corollary of Gauss's theorem, quoted from **[4**, p. 33], is useful (see also **[3**, p. 550]).

THEOREM 1. *It is impossible to construct with ruler and compasses a line whose length is a root or the negative of a root of a cubic equation with rational coefficients having no rational root, when the unit of length is given.*

Problem 115. Given G, H_a, H_b.

Particular positions are chosen for the located points, and it is shown that they determine a triangle for which there is no straightedge and compasses construction. Let the located points in a rectangular coordinate system be $G=(1,\frac{2}{3})$, $H_a=(0,1)$, and $H_b=(0,-1)$. (See FIGURE 5.)

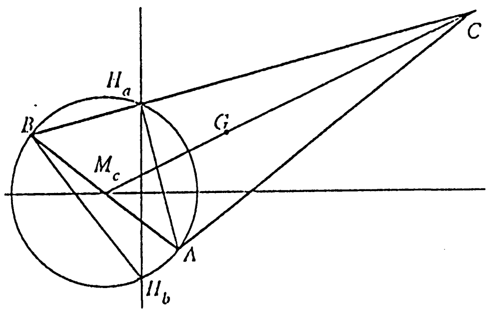


FIGURE 5

Since $\angle AH_aB = \angle AH_bB = 90°$, the points H_a and H_b lie on the circle having the segment AB as diameter and M_c as center. Then M_c lies on the perpendicular bisector of the segment H_aH_b. Hence $M_c=(x,0)$ for some real number x. Since G is $2/3$ of the way from C to M_c, we have $C=(3-2x,2)$. Suppose that $A=(u,v)$ for some real numbers u and v. Then $B=2M_c-A=(2x-u,-v)$. Since C, A, and H_b are collinear, the slopes of the lines AH_b and H_bC must be equal. Thus

$$\frac{v+1}{u}=\frac{3}{3-2x}.$$

Similarly, collinearity of C, B, and H_a yields

$$\frac{v+1}{u-2x}=\frac{1}{3-2x}.$$

If we divide the first equation by the second and solve for u, we obtain

$$u=-x \quad \text{and then} \quad v=\frac{x+3}{2x-3}.$$

Since M_c is the circumcenter of right triangle ABH_b, we have

$$x^2+1=(x-u)^2+v^2,$$

and so substitution with simplification yields

$$2x^3-6x^2+4x+3=0,$$

which has no rational root. Hence by the theorem quoted above, x is nonconstructible. There *is* a triangle having the given located points, with $x\approx-0.4311$ and the nonconstructible vertices $A\approx(0.4311,-0.6651)$, $B\approx(-1.2934,0.6651)$, and $C\approx(3.8623,2)$.

Readers are invited to fill in the blanks still remaining in Wernick's problem list. The following 20 problems are open:

77. O, H_a, T_b	109. M_a, H, T_b	118. G, H_a, T_b	127. H_a, H_b, T_c	135. H_a, T_b, I
78. O, H_a, I	110. M_a, H, I	119. G, H_a, I	128. H_a, H_b, I	136. H, T_a, T_b
81. O, T_a, T_b	111. M_a, T_a, T_b	122. G, T_a, T_b	132. H_a, T_a, T_b	137. H, T_a, I
90. M_a, M_b, I	113. M_a, T_b, T_c	123. G, T_a, I	134. H_a, T_b, T_c	138. T_a, T_b, T_c

Note. Not much seems to have been published on the construction of triangles using located points. Most works on geometric constructions, such as the excellent [**7**], treat triangle construction problems only from the point of view of "parts", such as sides, angles, medians, altitudes, and the like. A systematic list of "parts" problems, together with solutions to some of them, can be found in [**8**], and a smaller systematic list, together with solutions, is [**5**]; the corresponding unsolvable "parts" problems are treated in [**6**]. Recent "challenge" columns are [**1, 2**].

Acknowledgment. The author thanks William Wernick for his encouragement during the work leading to this paper.

REFERENCES

1. George Berzsenyi, Constructing triangles from three given parts, *Quantum* **4**, no. 6, July/Aug. 1994, 30, 55.
2. George Berzsenyi, Constructing triangles from three located points, *Quantum* **5**, no. 1, Sept./Oct. 1994, 54.
3. G. Chrystal, *Algebra, an Elementary Textbook*, 7th edition, Vol. 1, reprint by Chelsea Publishing Co., New York, 1964.
4. Leonard Eugene Dickson, *New First Course in the Theory of Equations*, John Wiley & Sons, Inc., New York, 1939.
5. Kurt Herterich, *Dreieckskonstruktionen*, Ernst Klett Verlag, Stuttgart, 1961; retitled *Die Konstruktion von Dreiecken*, 1986.
6. Otto Krötenheerdt, Zur Theorie der Dreieckskonstruktionen, *Wissenschaftliche Zeitschrift der Martin-Luther-Univ. Halle-Wittenberg*, Math.-Naturw. Reihe **15** (1966), 677–700.
7. Julius Petersen, *Methods and Theories for the Solution of Problems of Geometrical Constructions*, Copenhagen, 1879; reprint in *String Figures and Other Monographs* by Chelsea Publishing Co., New York, 1960.
8. Alfred S. Posamentier and William Wernick, *Geometric Constructions*, J. Weston Walch, Portland, ME, 1973; expanded edition, *Advanced Geometric Constructions*, Dale Seymour Publications, Palo Alto, CA, 1988.
9. William Wernick, Triangle constructions with three located points, this MAGAZINE **55** (1982), 227–230.

Ceva's and Menelaus' Theorems and Their Converses via Centroids

MURRAY S. KLAMKIN
University of Alberta
Edmonton, Alberta, Canada T6G 2G1

SIDNEY H. KUNG
Jacksonville University
Jacksonville, FL 32211

It appears that present-day students do not know much about applications of centroids to geometry. Perhaps this note may help rectify this situation. For further applications, see [**1**], [**2**], and [**4**].

Ceva's theorem states that if AD, BE, and CF are three concurrent cevians of a triangle ABC as in FIGURE 1, then

$$\left(\frac{BD}{DC}\right)\left(\frac{CE}{EA}\right)\left(\frac{AF}{FB}\right)=1. \tag{1}$$

Note. Not much seems to have been published on the construction of triangles using located points. Most works on geometric constructions, such as the excellent [**7**], treat triangle construction problems only from the point of view of "parts", such as sides, angles, medians, altitudes, and the like. A systematic list of "parts" problems, together with solutions to some of them, can be found in [**8**], and a smaller systematic list, together with solutions, is [**5**]; the corresponding unsolvable "parts" problems are treated in [**6**]. Recent "challenge" columns are [**1**, **2**].

Acknowledgment. The author thanks William Wernick for his encouragement during the work leading to this paper.

REFERENCES

1. George Berzsenyi, Constructing triangles from three given parts, *Quantum* **4**, no. 6, July/Aug. 1994, 30, 55.
2. George Berzsenyi, Constructing triangles from three located points, *Quantum* **5**, no. 1, Sept./Oct. 1994, 54.
3. G. Chrystal, *Algebra, an Elementary Textbook*, 7th edition, Vol. 1, reprint by Chelsea Publishing Co., New York, 1964.
4. Leonard Eugene Dickson, *New First Course in the Theory of Equations*, John Wiley & Sons, Inc., New York, 1939.
5. Kurt Herterich, *Dreieckskonstruktionen*, Ernst Klett Verlag, Stuttgart, 1961; retitled *Die Konstruktion von Dreiecken*, 1986.
6. Otto Krötenheerdt, Zur Theorie der Dreieckskonstruktionen, *Wissenschaftliche Zeitschrift der Martin-Luther-Univ. Halle-Wittenberg*, Math.-Naturw. Reihe **15** (1966), 677–700.
7. Julius Petersen, *Methods and Theories for the Solution of Problems of Geometrical Constructions*, Copenhagen, 1879; reprint in *String Figures and Other Monographs* by Chelsea Publishing Co., New York, 1960.
8. Alfred S. Posamentier and William Wernick, *Geometric Constructions*, J. Weston Walch, Portland, ME, 1973; expanded edition, *Advanced Geometric Constructions*, Dale Seymour Publications, Palo Alto, CA, 1988.
9. William Wernick, Triangle constructions with three located points, this MAGAZINE **55** (1982), 227–230.

Ceva's and Menelaus' Theorems and Their Converses via Centroids

MURRAY S. KLAMKIN
University of Alberta
Edmonton, Alberta, Canada T6G 2G1

SIDNEY H. KUNG
Jacksonville University
Jacksonville, FL 32211

It appears that present-day students do not know much about applications of centroids to geometry. Perhaps this note may help rectify this situation. For further applications, see [**1**], [**2**], and [**4**].

Ceva's theorem states that if AD, BE, and CF are three concurrent cevians of a triangle ABC as in FIGURE 1, then

$$\left(\frac{BD}{DC}\right)\left(\frac{CE}{EA}\right)\left(\frac{AF}{FB}\right) = 1. \tag{1}$$

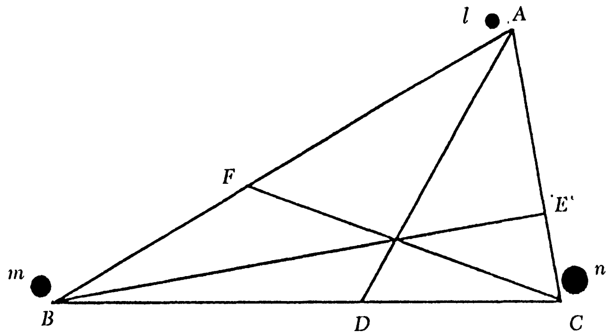


FIGURE 1

Let $BF/FA = l/m$ and $AE/EC = n/l$. We now place masses l, m, n, at A, B, and C, respectively. We can replace masses l and m by one mass $l+m$ at F (since F is their centroid). Hence the centroid of all the masses lies on line CF. Similarly, the centroid lies on line BE and thus must be at the intersection of BE and CF. It now follows that the centroid of masses m and n (in FIGURE 1) must be at D so that $CD/DB = m/n$ giving (1).

Converse (useful in proving concurrency results):

If D, E, F are points on the sides BC, CA, and AB of a triangle ABC such that

$$\left(\frac{BD}{DC}\right)\left(\frac{CE}{EA}\right)\left(\frac{AF}{FB}\right) = 1,$$

then AD, BE, and CF are concurrent. As before, referring to FIGURE 1, the centroid of the three masses must lie on each of the lines AD, BE, and CF so that these lines must be concurrent.

Menelaus' theorem states that if X, Y, Z are collinear points on the sides of a triangle ABC as in FIGURE 2, then

$$\left(\frac{AZ}{ZB}\right)\left(\frac{BX}{XC}\right)\left(\frac{CY}{YA}\right) = -1. \tag{2}$$

Let $AZ/ZB = m/l$ and $BX/XC = -n/m$. We now place masses $l+m$ at Z, $l+n$ at Y and $n-m$ at X as in FIGURE 3.

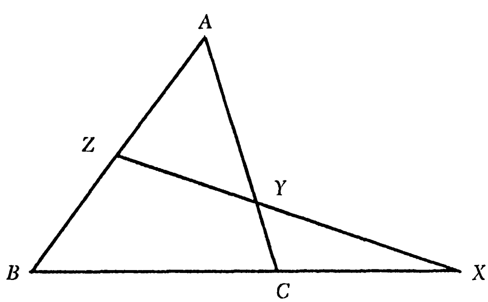


FIGURE 2

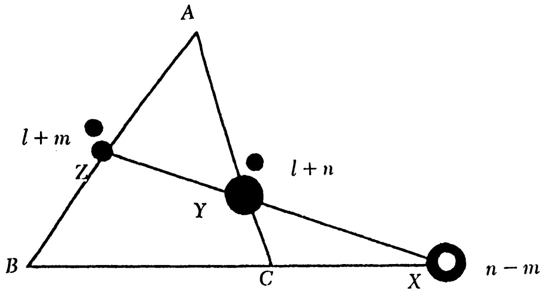


FIGURE 3

The centroid of the three sets of masses lies on the line ZYX. We can replace the masses $l+m$ at Z by masses l at A and m at B. Now we can replace the masses m at B and $n-m$ at X by n at C as in FIGURE 4. Thus the centroid of the masses must also lie on line AC and consequently must be at Y. Hence, $CY/YA = l/n$, which gives (2).

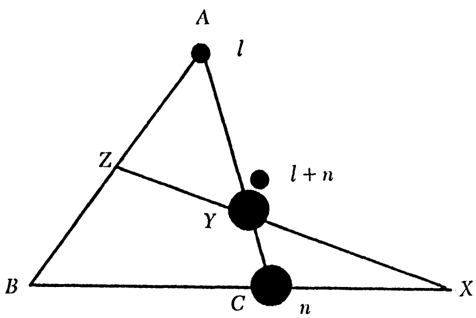


FIGURE 4

Converse (useful in proving collinearity results):

If points X, Y, Z are taken on the sides of ABC as in FIGURE 2 such that $AZ/ZB = m/l$, $BX/XC = -n/m$, and $CY/YA = l/n$, then X, Y, Z are collinear. We proceed as before and obtain FIGURE 4. Since here $CY/YA = l/n$, it follows that Y is the centroid of all the masses. This implies that X, Y, Z are collinear.

In [4], a simultaneous generalization is given of the theorems of Ceva and Menelaus. It would be of interest to give a centroid proof of this result.

REFERENCES

1. C. V. Durell, *Modern Geometry*, Macmillan, London, 1952, pp. 49–64.
2. M. S. Klamkin, Vector proofs in solid geometry, *Amer. Math. Monthly* **77** (1970), 1059.
3. M. S. Klamkin and Andy Liu, Simultaneous generalization of the theorems of Ceva and Menelaus, this MAGAZINE, **65** (1992), 48–52.
4. V. A. Uspenskii, *Some Applications of Mechanics to Mathematics*, Blaisdell, London, 1961.

A Short Solution of a Problem in Combinatorial Geometry

MARC NOY
Dept. Matemàtica Aplicada II
Universitat Politècnica de Catalunya
08028 Barcelona, Spain

The solution of the classical problem: Into how many regions is a circle divided if n points on its circumference are joined by chords with no three concurrent? is well known to be

$$\binom{n}{4}+\binom{n}{2}+1$$

and can be found in many references. It appears in [2] as problem 47, although the two solutions provided are quite involved. Because the first values of n yield 1, 2, 4, 8, 16, and the next one 31, it appears in [1] as one of several examples of patterns that seem to appear in a sequence of numbers, but turn out not to be correct. The solution there considers the circle as a planar map with $V=n+\binom{n}{4}$ vertices—the original n points and an intersection of chords for each choice of 4 of those points. There are 4 ends of edges at each intersection and $n+1$ at each point, so that $E=\frac{1}{2}n(n+1)+2\binom{n}{4}$ and Euler's formula $R=E-V+1$ gives the result.

In this note we offer a direct combinatorial proof without any algebraic calculation. Imagine that we draw the chords one after another, keeping count of the number of regions created each time. Starting with one region, the whole circle, the first chord creates one more region. Any subsequent chord will create one new region, plus as many more as the number of intersections it produces with the chords previously

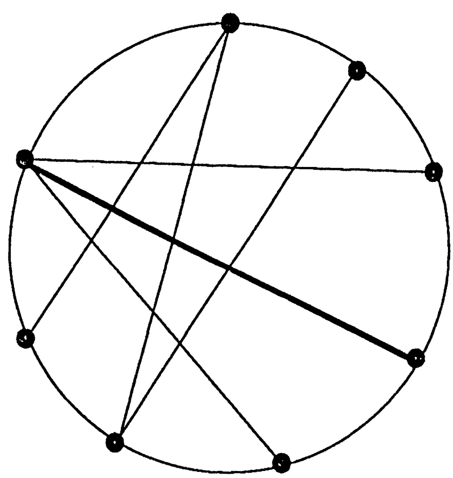


FIGURE: Adding the thick chord creates 4 new regions.

drawn (see FIGURE). This number will of course depend on the particular chord selected but at the end there will be one new region for every chord and one for every intersection. The number of chords is $\binom{n}{2}$ and the number of intersections is $\binom{n}{4}$ as before. Hence the number of regions is $1+\binom{n}{2}+\binom{n}{4}$.

REFERENCES

1. R. K. Guy, The strong law of small numbers, *Amer. Math. Monthly* (1988), 697–712.
2. A. M. Yaglom and I. M. Yaglom, *Challenging Mathematical Problems with Elementary Solutions*, Vol. I, Dover Publications, Mineola, NY, 1987.

On the Order of a Product in a Finite Abelian Group

DIETER JUNGNICKEL
Universität Augsburg
D-86135 Augsburg, F.R. Germany

The following result from elementary group theory is included in virtually every course in abstract algebra:

LEMMA 1. *Let a and b be two elements of a finite abelian group G with orders m and n, respectively. If m and n are co-prime, then the product ab has order mn.*

Lemma 1 can be used to find elements of increasingly larger order in G. This has many interesting applications, both theoretic and algorithmic. One usually applies Lemma 1 to show that G is cyclic if, and only if, its exponent agrees with its order; this result in turn is used to show that a finite subgroup of the multiplicative group of a field is cyclic. (See, e.g., Jacobson [**1**, Theorems 1.4 and 2.18] or van der Waerden [**5**, §§42 and 43].) Lemma 1 is also the basis of the standard algorithm (due to Gauss) for determining primitive elements for finite fields (i.e., generators for the multiplicative groups) and then primitive polynomials, see e.g. Jungnickel [**3**, §2.5]. These are important tasks if one actually wants to perform arithmetic in finite fields, which in turn is fundamental for applications, e.g. in cryptography. See [**3**] (and the references cited there) for more information on this topic.

In some of my algebra classes, students asked the quite natural question: What happens in the situation of Lemma 1 if one drops the hypothesis that m and n are co-prime.* Trivially, $(ab)^{\mathrm{lcm}(m,n)}=1$, so that the order of ab satisfies

$$\mathrm{o}(ab) \mid \mathrm{lcm}(m,n). \qquad (1)$$

Usually, some student suggests that one should actually have equality in (1). Though this might seem a reasonable conjecture, it is not difficult to find counterexamples. Here is a simple series of such examples.

*I do not know of any textbook treating this question. Weak versions of some of the results below (i.e., Corollary 1, the special case $f=d$ of Lemma 3 and the corresponding construction) were already obtained in [**2**].

drawn (see FIGURE). This number will of course depend on the particular chord selected but at the end there will be one new region for every chord and one for every intersection. The number of chords is $\binom{n}{2}$ and the number of intersections is $\binom{n}{4}$ as before. Hence the number of regions is $1+\binom{n}{2}+\binom{n}{4}$.

REFERENCES

1. R. K. Guy, The strong law of small numbers, *Amer. Math. Monthly* (1988), 697–712.
2. A. M. Yaglom and I. M. Yaglom, *Challenging Mathematical Problems with Elementary Solutions*, Vol. I, Dover Publications, Mineola, NY, 1987.

On the Order of a Product in a Finite Abelian Group

DIETER JUNGNICKEL
Universität Augsburg
D-86135 Augsburg, F.R. Germany

The following result from elementary group theory is included in virtually every course in abstract algebra:

LEMMA 1. *Let a and b be two elements of a finite abelian group G with orders m and n, respectively. If m and n are co-prime, then the product ab has order mn.*

Lemma 1 can be used to find elements of increasingly larger order in G. This has many interesting applications, both theoretic and algorithmic. One usually applies Lemma 1 to show that G is cyclic if, and only if, its exponent agrees with its order; this result in turn is used to show that a finite subgroup of the multiplicative group of a field is cyclic. (See, e.g., Jacobson [**1**, Theorems 1.4 and 2.18] or van der Waerden [**5**, §§42 and 43].) Lemma 1 is also the basis of the standard algorithm (due to Gauss) for determining primitive elements for finite fields (i.e., generators for the multiplicative groups) and then primitive polynomials, see e.g. Jungnickel [**3**, §2.5]. These are important tasks if one actually wants to perform arithmetic in finite fields, which in turn is fundamental for applications, e.g. in cryptography. See [**3**] (and the references cited there) for more information on this topic.

In some of my algebra classes, students asked the quite natural question: What happens in the situation of Lemma 1 if one drops the hypothesis that m and n are co-prime.* Trivially, $(ab)^{\operatorname{lcm}(m,n)}=1$, so that the order of ab satisfies

$$o(ab)\mid \operatorname{lcm}(m,n). \tag{1}$$

Usually, some student suggests that one should actually have equality in (1). Though this might seem a reasonable conjecture, it is not difficult to find counterexamples. Here is a simple series of such examples.

*I do not know of any textbook treating this question. Weak versions of some of the results below (i.e., Corollary 1, the special case $f=d$ of Lemma 3 and the corresponding construction) were already obtained in [**2**].

Example 1 Let x generate a cyclic group of order $2pq$, where p and q are distinct odd primes, and put $a := x^q$ and $b := x^p$. Then we have $m = 2p$ and $n = 2q$, hence $\operatorname{lcm}(m, n) = 2pq$. But $ab = x^{p+q}$ has order at most (in fact, exactly) pq, since $p + q$ is even.

If one examines Example 1 more closely, one realizes that the groups A and B generated by a and b, respectively, intersect in a subgroup of order 2. Might this be the reason why the order of ab differs from $\operatorname{lcm}(a, b)$ by a factor of 2? As we shall see in the following example, the situation is not quite that simple. (This example can actually be generalized considerably, cf. Theorem 2 below.) Nevertheless, the size of the intersection of A and B indeed plays an important role, as Lemma 2 and Theorem 1 below will indicate.

Example 2 Choose three distinct primes p, q, r and assume $p < q < r$. Let G be a cyclic group of order pqr, generated by the element g, and put $a := g^q$ and $b := g^p$. Then $m = pr$, $n = qr$ and $A \cap B$ has order r; here $ab = g^{p+q}$ has order pqr (since pqr is co-prime with $p + q$). Now select the smallest positive integer x for which $q + xp \equiv 0 \pmod r$ and replace b by $b' := g^{xp}$. Note that

$$q + xp \leq q + (r-1)p < rq;$$

thus qr cannot divide $q + xp$, and therefore q cannot divide x. Hence b' also has order n, but now ab' has order pq.

LEMMA 2. *Let a and b be two elements of a finite abelian group G with orders m and n, respectively, and put $d := \gcd(m, n)$. Denote the subgroups of G generated by a and b by A and B, respectively, and assume that $A \cap B$ has order s (where s divides d). Then the order of the product ab satisfies the condition*

$$\frac{mn}{ds} \mid o(ab) \mid \frac{mn}{d} = \operatorname{lcm}(m, n). \tag{2}$$

Proof. Write $k := o(ab)$. As noted in (1), k divides $\operatorname{lcm}(m, n)$. Now observe that $(ab)^k = 1$ implies

$$a^k = b^{-k} =: c \in A \cap B. \tag{3}$$

By hypothesis, $A \cap B$ is a cyclic group of order s. Since the unique subgroup of A of order s is generated by $a^{m/s}$, we have $c = a^{xm/s}$ for some positive integer x. In view of (3) this implies

$$k \equiv x\frac{m}{s} \pmod m,$$

and hence m/s divides k. A similar argument (applied to B) shows that n/s also divides k, and therefore $\operatorname{lcm}(m/s, n/s)$ divides k. Using $\gcd(m/s, n/s) = d/s$, one sees that $\operatorname{lcm}(m/s, n/s) = mn/ds$.

COROLLARY 1. *Let a and b be two elements of a finite abelian group G with orders m and n, respectively, and put $d := \gcd(m, n)$. Then the order of the product ab satisfies the condition*

$$\frac{mn}{d^2} \mid o(ab) \mid \frac{mn}{d} = \operatorname{lcm}(m, n). \tag{4}$$

As we will see next, not every integer allowed by the preceding conditions can actually occur as the order of the product ab for a suitable choice of a and b. In the situation of Lemma 2, we have $o(ab) = mn/df$ for some divisor f of s. We now obtain a further condition on f.

LEMMA 3. *Under the assumptions of Lemma 2, write $o(ab) = mn/df$ for some divisor f of s. Then f satisfies the condition*

$$\gcd(f, m/d) = \gcd(f, n/d) = 1. \tag{5}$$

Proof. We analyze the situation encountered in the proof of Lemma 2 more closely (using the hypothesis $k = mn/df$) and claim that the element c defined in (3) has order f. To see this, write $e' := o(c)$ and note first

$$c^f = a^{kf} = a^{mn/d} = (a^m)^{n/d} = 1,$$

so that e' divides f. On the other hand, c actually belongs to the unique subgroup of order e' of $A \cap B$, and the same argument as in the proof of Lemma 2 (now applied to e' instead of s) shows that $\operatorname{lcm}(m/e', n/e') = mn/de'$ divides $k = mn/df$. Therefore f has to divide e', and we indeed obtain $e' = f$. But the elements of A of order f are precisely the elements of the form $a^{xm/f}$ for some positive integer x that is co-prime with f. Hence we have

$$c = a^k = a^{xm/f} \quad \text{with } \gcd(x, f) = 1,$$

and thus $mn/df \equiv xm/f \pmod m$. This implies $n/d \equiv x \pmod f$, and therefore indeed $\gcd(n/d, f) = 1$. Similarly, we also obtain $\gcd(m/d, f) = 1$.

We can now give the following improved version of Lemma 2.

THEOREM 1. *Let a and b be two elements of a finite abelian group G with orders m and n, respectively, and put $d := \gcd(m, n)$. Denote the subgroups of G generated by a and b by A and B, respectively, and assume that $A \cap B$ has order s (where s divides d). Let e be the largest divisor of s satisfying condition* (5) *above. Then the order of the product ab satisfies*

$$\frac{mn}{de} \,\Big|\, o(ab) \,\Big|\, \frac{mn}{d\varepsilon}, \tag{6}$$

where $\varepsilon = 1$ if e is odd and $\varepsilon = 2$ otherwise.

Proof. By Lemma 2, $o(ab) = mn/df$ for some divisor f of s. Since f has to satisfy condition (5), one immediately concludes that $o(ab)$ must be a multiple of mn/de, where e is the largest divisor satisfying condition (5). Regarding the upper bound in (6), we already know that $o(ab)$ divides mn/d. Now assume that e is even; then s and mn/d are likewise even. Since e satisfies condition (5), m/d and n/d are odd. So $a^{mn/2d} = b^{mn/2d}$ is the unique involution in $A \cap B$. This implies $(ab)^{mn/2d} = 1$, proving the assertion.

An algorithm for determining the integer e defined in Theorem 1 can be found in Lüneburg [4, Ch. IV]. We will soon see that Theorem 1 is best possible (for every choice of m, n and s): Both the lower and the upper bound can always be realized. To this purpose we require the following simple auxiliary result that shows that every invertible residue modulo m can be "lifted" to an invertible residue modulo n for an arbitrary multiple n of m.

LEMMA 4. *Let m be a positive integer, let n be any multiple of m, and let α be an arbitrary integer that is invertible modulo m. Then there exists an integer β that is invertible modulo n and satisfies $\alpha \equiv \beta \pmod m$.*

Proof. It clearly suffices to consider the case where $n = mp$ for some prime p. By hypothesis, $\gcd(\alpha, m) = 1$; we need to determine an integer β with $\gcd(\beta, mp) = 1$ and $\alpha \equiv \beta \pmod{m}$. If p divides m or if p does not divide α, we may simply take $\beta := \alpha$. Thus assume that p divides α, but not m. Then we can choose $\beta := \alpha + m$.

THEOREM 2. *Let m, n and s be arbitrary positive integers for which s divides both m and n. Then there exists a finite abelian group G with cyclic subgroups A and B of orders m and n, respectively. $A \cap B$ is a group of sizes and the generators a, a' for A and b, b' for B satisfy*

$$o(ab) = \frac{mn}{de} \quad \textit{and} \quad o(a'b') = \frac{mn}{d\varepsilon}, \tag{7}$$

where d, e and ε are defined as in Theorem 1.

Proof. Let G be the finite abelian group generated by two elements x and y of order m and n, respectively, satisfying the relation $x^{m/s} = y^{n/s} =: c$. (For an explicit construction of this group, see the remark following this proof.) Let A and B be the subgroups of G generated by x and y, respectively. Then the generators of A are precisely the elements of the form $a := x^{\alpha}$, where α is a positive integer satisfying $\gcd(\alpha, m) = 1$; similarly, the generators of B are precisely the elements of the form $b := y^{\beta}$, where β is a positive integer satisfying $\gcd(\beta, n) = 1$. By Theorem 1, mn/de divides $o(ab)$ for every choice of α and β. We first want to select α and β in such a way that equality holds. We compute

$$(ab)^{mn/de} = \left(x^{m/e}\right)^{\alpha n/d} \cdot \left(y^{n/e}\right)^{\beta m/d} = \left(c^{s/e}\right)^{(\alpha n + \beta m)/d}. \tag{8}$$

By hypothesis, $\gcd(e, m/d) = \gcd(e, n/d) = 1$. Thus we can choose positive integers α' and β' with

$$\alpha' \frac{n}{d} \equiv 1 \pmod{e} \quad \text{and} \quad \beta' \frac{m}{d} \equiv -1 \pmod{e}.$$

Since e is a common divisor of m and n, Lemma 4 guarantees the existence of positive integers α and β satisfying

$$\alpha \equiv \alpha' \pmod{e}, \quad \beta \equiv \beta' \pmod{e} \quad \text{and} \quad \gcd(\alpha, m) = \gcd(\beta, n) = 1.$$

With this choice of α and β, we have

$$(\alpha n + \beta m)/d \equiv (\alpha' n + \beta' m)/d \equiv 0 \pmod{e},$$

and thus (8) yields the desired conclusion $(ab)^{mn/de} = 1$, since $c^{s/e}$ has order e. Finally, generators a' and b' satisfying $o(a'b') = mn/d\varepsilon$ can be determined in an analogous way by selecting α' and β' with

$$\alpha' \frac{n}{d} \equiv \beta' \frac{m}{d} \equiv 1 \pmod{e}.$$

One then obtains

$$(a'b')^{mn/de} = \left(c^{s/e}\right)^{(\alpha n + \beta m)/d} = \left(c^{s/e}\right)^{2}.$$

This shows $o(a'b') = (mn/de) \cdot o(c^{2s/e})$, which easily gives the assertion.

We remark that the group G used in the proof of Theorem 2 is, of course, uniquely determined by the properties required there. An explicit construction can be given as follows. Let v and w be generators for cyclic groups of orders m and n, respectively, put $H := \langle v\rangle \times \langle w\rangle$, and note that $U := \langle (v^{m/s}, w^{n/s})\rangle$ is a subgroup of order s of H. Now define

$$G := H/U, \quad x := (v,1)U \quad \text{and} \quad y := (1, w^{-1})U.$$

Then x and y generate groups of orders m and n, respectively, which intersect in a subgroup of order s (generated by $x^{m/s} = y^{n/s} =: c$), since one has

$$x^{m/s}(y^{-1})^{n/s} = (v^{m/s},1)U\cdot(1,w^{n/s})U = (v^{m/s}, w^{n/s})U = U.$$

As a consequence of Theorem 2, we obtain the following result, which gives a complete answer to our original question for all choices of m and n.

THEOREM 3. *Let m and n be arbitrary positive integers, put $d := \gcd(m,n)$, and let k be any positive integer satisfying*

$$\frac{mn}{df} \,|\, k \,|\, \frac{mn}{d}, \tag{9}$$

where f is the largest divisor of d for which one has

$$\gcd(f, m/d) = \gcd(f, n/d) = 1. \tag{10}$$

Then there exist a finite abelian group G and elements a and b of G with orders m and n, respectively, such that $o(ab) = k$.

Proof. Note that $k = mn/ds$ for some divisor s of f, and apply Theorem 2 with this choice of s. Because of condition (10), one has $e = s$, and thus the first case of (7) gives the assertion.

REFERENCES

1. N. L. Jacobson, *Basic Algebra I, 2nd edition*, Freeman, San Francisco, 1985.
2. D. Jungnickel, Eine Bemerkung über endliche abelsche Gruppen, *Math. Semesterber.* **34** (1987), 116–120.
3. D. Jungnickel, *Finite fields. Structure and arithmetics*, Bibliographisches Institut, Mannheim, 1993.
4. H. Lüneburg, *On the rational normal form of endomorphisms: A primer to constructive algebra*, Bibliographisches Institut, Mannheim, 1987.
5. B. L. van der Waerden, *Algebra I (8. Auflage)*, Springer, Berlin, 1971.

On Copying a Compact Disk to Cassette Tape: An Integer-Programming Approach

SAUL I. GASS
College of Business and Management
University of Maryland
College Park, MD 20742

1. The musical tape It is always rewarding when a question from a student challenges both the instructor and the class. During a lecture on model formulation using integer variables, a student inquired as to whether there was a model and put $H := \langle v \rangle \times \langle w \rangle$, and note that $U := \langle (v^{m/s}, w^{n/s}) \rangle$ is a subgroup of order s of H. Now define

$$G := H/U, \quad x := (v,1)U \quad \text{and} \quad y := (1, w^{-1})U.$$

Then x and y generate groups of orders m and n, respectively, which intersect in a subgroup of order s (generated by $x^{m/s} = y^{n/s} =: c$), since one has

$$x^{m/s}(y^{-1})^{n/s} = (v^{m/s},1)U \cdot (1, w^{n/s})U = (v^{m/s}, w^{n/s})U = U.$$

As a consequence of Theorem 2, we obtain the following result, which gives a complete answer to our original question for all choices of m and n.

THEOREM 3. *Let m and n be arbitrary positive integers, put $d := \gcd(m,n)$, and let k be any positive integer satisfying*

$$\frac{mn}{df} \mid k \mid \frac{mn}{d}, \tag{9}$$

where f is the largest divisor of d for which one has

$$\gcd(f, m/d) = \gcd(f, n/d) = 1. \tag{10}$$

Then there exist a finite abelian group G and elements a and b of G with orders m and n, respectively, such that $o(ab) = k$.

Proof. Note that $k = mn/ds$ for some divisor s of f, and apply Theorem 2 with this choice of s. Because of condition (10), one has $e = s$, and thus the first case of (7) gives the assertion.

REFERENCES

1. N. L. Jacobson, *Basic Algebra I, 2nd edition*, Freeman, San Francisco, 1985.
2. D. Jungnickel, Eine Bemerkung über endliche abelsche Gruppen, *Math. Semesterber.* **34** (1987), 116–120.
3. D. Jungnickel, *Finite fields. Structure and arithmetics*, Bibliographisches Institut, Mannheim, 1993.
4. H. Lüneburg, *On the rational normal form of endomorphisms: A primer to constructive algebra*, Bibliographisches Institut, Mannheim, 1987.
5. B. L. van der Waerden, *Algebra I (8. Auflage)*, Springer, Berlin, 1971.

On Copying a Compact Disk to Cassette Tape: An Integer-Programming Approach

SAUL I. GASS
College of Business and Management
University of Maryland
College Park, MD 20742

1. The musical tape It is always rewarding when a question from a student challenges both the instructor and the class. During a lecture on model formulation using integer variables, a student inquired as to whether there was a model and algorithm for accomplishing the following. He often records the musical numbers from a compact disk (CD) onto a two-sided cassette tape. He wondered if there were a way he could divide the musical numbers into two subcollections, one for side A and one for side B, so that the difference in the times it took to play each side was as small as possible. He would really like the playing times to be equal, but recognized that would often not be possible.

This challenge led us to an easily understood integer-programming formulation of the problem. I believe that this problem, which also appears in other contexts, is a natural one for introducing some basic mathematical programming concepts to mathematics students.

2. Variations on a theme The tape recording problem and various equivalent statements are as follows:

(0) Given N songs with the playing time for the ith song denoted by t_i, divide the songs into two sets A and B such that the total playing time for the songs in set A and the total playing time for the songs in set B are as close as possible.

(1) Given a set of $N \geq 2$ positive numbers n_i, divide them into two nonempty subsets A and B such that the absolute difference between the sums of the numbers in each set is minimized.

(2) Given two similar machines $M1$ and $M2$, and N tasks such that the ith task can be processed on either $M1$ or $M2$ in the same time t_i, divide the tasks between the two machines and let $A = \Sigma t_i$ (for tasks assigned to $M1$) and let $B = \Sigma t_i$ (for tasks assigned to $M2$) so that if the machines start processing the tasks at the same time and continue without any delays, then all the tasks are finished in the minimum amount of time.

(3) Given N packages with weights w_i, divide the packages between two delivery trucks so that the total weight A on the first truck is as close as possible to the total weight B on the second truck.

Versions (0), (1), and (3) are basically problems in combinatorics and optimization, while version (2) is a well-studied optimization problem in scheduling [**1**, **2**, **3**]. Other statements of the problem include the division of typing tasks between equally proficient typists to minimize the time for completing all the tasks, the scheduling of jobs on computers, or the scheduling of jobs on copying machines [**1**, **3**, **4**]. It is clear that versions (0), (1), and (3) are equivalent. Their equivalence to (2) is seen by noting that since $A + B$ is fixed, and since we want to minimize the larger of A and B, we must have A and B as close as possible.

3. Finding a tune to play There are no known efficient (polynomial time) algorithms to solve these equivalent problems. The problems are *NP*-hard, and known methods for finding general solutions to such problems may blow up exponentially, [**3**, **4**, **5**]. We first discuss solving the problem in terms of version (1).

Various heuristic algorithms can be devised to solve the problem, with the simplest being a greedy-type algorithm that can be described as follows.

Largest Processing-Time (LPT) Algorithm Order the numbers in decreasing order of their size so that $n_1 \geq n_2 \geq \cdots \geq n_N$. Let $S(A)$ be the sum of the numbers in subset A and $S(B)$ the sum of the numbers in subset B, with these numbers updated as numbers are placed into the subsets. We start with $S(A) = S(B) = 0$. Place n_1 in A and n_2 in B. The next number is placed into the subset that has the smallest sum, with ties broken arbitrarily. The process is repeated until all numbers are in a subset.

The measure of success of the algorithm is the difference of the final sums, that is, $D = |S(A) - S(B)|$. The best D can be is 0 if Σn_i is even or 1 if this sum is odd. For example, if the numbers were $8, 7, 6, 5, 4, 3, 2, 1$, we would have $A = (8, 5, 4, 1)$ and $B = (7, 6, 3, 2)$, with $S(A) = 18$, $S(B) = 18$, and $D = 0$. The LPT algorithm (also termed the decreasing-time-list algorithm) was first developed and applied to scheduling problems by Graham [3]. He showed that the algorithm, as applied here for two subsets, will produce an $S(A)$ and an $S(B)$ such that the larger of the two will never be more than 17% from the value it would have in an optimal placement. For the numbers $11, 10, 9, 6, 4$, the algorithm produces $A = (11, 6, 4)$, $B = (10, 9)$, with $S(A) = 21$, $S(B) = 19$, and $D = 2$. The optimal placements are $A = (11, 9)$, $B = (10, 6, 4)$, with $S(A) = S(B) = 20$, $D = 0$ [**1**, p. 75].

For N large, determining the optimal solution by enumerating all possible combinations is a rather difficult and tedious job. Not all combinations need to be evaluated, however. If Σn_i is odd, then only the first $(2^{N-1} - 1)$ combinations need to be evaluated due to the symmetry of the combinatorial terms. (The combinations $\binom{N}{0}$ and $\binom{N}{N}$ are ignored and we need half of the other combinations as $\binom{N}{m} = \binom{N}{N-m}$.) If N is even, $1/2\binom{N}{N/2}$ additional combinations must be evaluated. For CDs, which usually have about 10 selections, one could use a computer to evaluate all combinations.

4. The musical mathematical-programming formulation We next discuss the problem in terms of the song-recording version Problem (0) and develop an integer-programming formulation that will be used to solve some numerical examples.

Mathematical Model (Integer-Programming Model)

Data

N = the total number of songs to be recorded

t_i = the time in seconds to play song i

Variables

$x_i = 1$ if song i is recorded on side A
$\quad\;\; 0$ if song i is not recorded on side A
$y_i = 1$ if song i is recorded on side B
$\quad\;\; 0$ if song i is not recorded on side B

$X = \Sigma t_i x_i$ = Total time on side A

$Y = \Sigma t_i y_i$ = Total time on side B

$D = X - Y$ = Difference between the total times on side A and side B

$D = D_1 - D_2 =$ The representation of the difference variable as the difference of two nonnegative variables; $D = D_1$ if $X > Y$; $D = -D_2$ if $X < Y$; $D = 0$ if $X = Y$.

Constraints

(1) $X = \Sigma t_i x_i$

(2) $Y = \Sigma t_i y_i$

(3) $X - Y = D_1 - D_2$

(4) $x_i + y_i = 1$ $(i = 1, \ldots, N)$

(5) $x_i = 0$ or 1

(6) $y_i = 0$ or 1

(7) $D_1 \geq 0,\ D_2 \geq 0$

Objective Function

Minimize $D_1 + D_2$

Constraints (1) and (2) are definitional constraints that define the total time on side A by X and the total time on side B by Y. These constraints and variables are not essential to the formulation, but are included as they explicitly return the total times for the songs on each side. Constraint (3) measures the absolute difference between X and Y by the difference between the non-negative variables D_1 and D_2; this eliminates the need to use the unrestricted variable D. Constraints (4), (5), and (6) force each song to be on either side A or side B. Constraints (7) restrict the difference variables to be nonnegative. The objective function seeks to minimize the sum of the difference variables. With this objective function, the optimal solution cannot have both D_1 and D_2 positive (otherwise you can show that the solution is not optimal). Both D_1 and D_2 can be zero, however.

5. A Stan Getz example Consider the following eight songs contained on the CD disc *Apasionado* by Stan Getz (A&M Records).

1. *Apasionado*	8:05 = 485 seconds	(x_1 or y_1)
2. *Coba*	7:05 = 425 seconds	(x_2 or y_2)
3. *Waltz for Stan*	6:05 = 365 seconds	(x_3 or y_3)
4. *Espanola*	4:15 = 255 seconds	(x_4 or y_4)
5. *Madrugada*	5:26 = 326 seconds	(x_5 or y_5)
6. *Amorous Cat*	4:58 = 298 seconds	(x_6 or y_6)
7. *Midnight Ride*	8:58 = 538 seconds	(x_7 or y_7)
8. *Lonely Lady*	5:39 = 339 seconds	(x_8 or y_8)

Integer-Programming Model

Minimize $D_1 + D_2$

subject to

$$X - 485x_1 - 425x_2 - 365x_3 - 255x_4 - 326x_5 - 298x_6 - 538x_7 - 339x_8 = 0$$
$$Y - 485y_1 - 425y_2 - 365y_3 - 255y_4 - 326y_5 - 298y_6 - 538y_7 - 339y_8 = 0$$
$$X - Y - D_1 + D_2 = 0$$
$$x_i + y_i = 1 \quad \text{for } i = 1, 2, \dots, 8$$
$$x_i = 0 \text{ or } 1$$
$$y_i = 0 \text{ or } 1$$
$$D_1 \geq 0, \ D_2 \geq 0$$

The optimal solution using a combined simplex algorithm and branch-and-bound procedure as implemented in the LINDO integer-programming software [6] is the following:

Side A—Time		*Side B*—Time		
$x_2 = 1$	425	$y_1 = 1$	485	
$x_4 = 1$	255	$y_3 = 1$	365	
$x_6 = 1$	298	$y_5 = 1$	326	
$x_7 = 1$	538	$y_8 = 1$	339	
TOTAL TIME:	1516		1515	$(D_1 - D_2) = 1$

It is interesting to note that the cassette version of the CD put out by A&M Records has songs 1, 2, 3, 4 on side A for a total time of 1530, and songs 5, 6, 7, 8 on side B for a total time of 1501, with a difference of 29 seconds! The largest-processing-time algorithm would put songs 3, 4, 7, 8 with time 1497 on side A and songs 1, 2, 5, 6 with time 1534 on side B for a difference of 37 seconds.

The recording company's ordering of the songs on a CD or tape may be due to artistic considerations or one may just want to have specific songs together. It is a simple matter, for example, to force *Apasionado* (song 1) and *Midnight Ride* (song 7) to be on the same side (and to follow one another) by setting $y_1 = 0$ and $y_7 = 0$ in the integer-programming formulation.

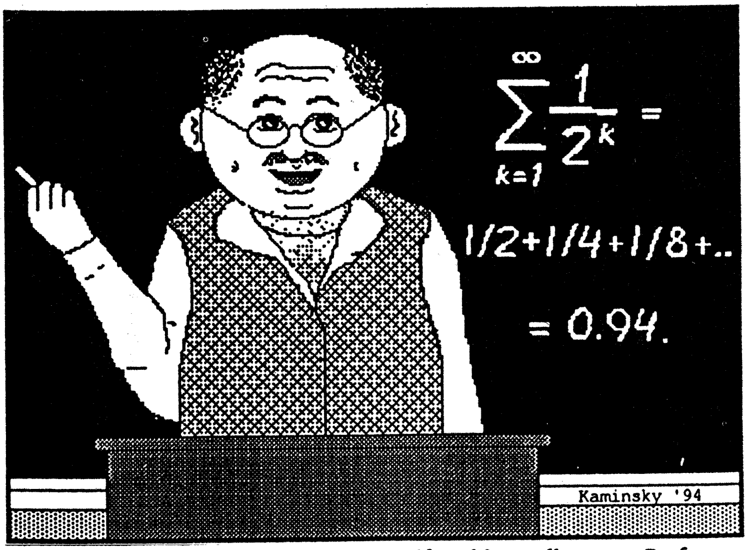
Acknowledgement. We wish to thank the anonymous referees for their constructive comments.

REFERENCES

1. COMAP, *For All Practical Purposes*, 2nd edition, W. H. Freeman and Co., New York, 1991.
2. M. R. Garey, R. L. Graham, and D. S. Johnson, Performance guarantees for scheduling algorithms, *Operations Research*, **26** (1978), 3–21.
3. R. L. Graham, Bounds on multiprocessing timing anomalies, *Siam J. Appl. Math.*, **17** (1969), 416–429.
4. R. L. Graham, *Mathematics Today: Twelve Informal Essays*, L. A. Steen, ed., Springer-Verlag, New York, 1978, 183–211.
5. S. K. Sahni, Algorithms for scheduling independent tasks, *J. Assoc. for Computing Machinery*, **23** (1976), 116–127.
6. L. Schrage, *LINDO*, 4th edition, The Scientific Press, South San Francisco, 1991.

Professor Fogelfroe

Professor F. Fogelfroe is Professor of Mathematics at ValuPak™ University, in Margo's Forehead, Minnesota.



In an effort to win a 6% raise for himself and his colleagues, Professor Fogelfroe carries through on his threat to reduce the accuracy of his lectures by a like amount.

—Kenneth Kaminsky
Augsburg College
Minneapolis, MN 55454

Chanson Sans Paroles

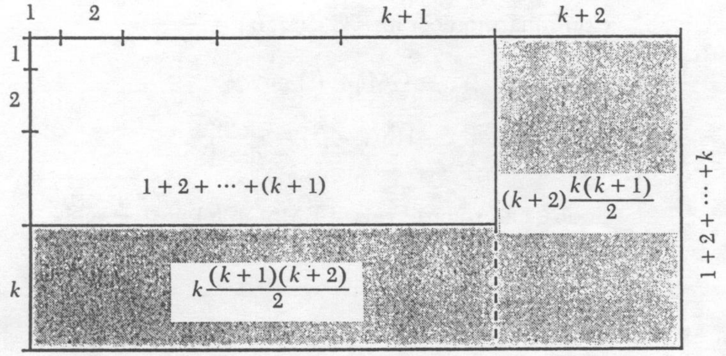
$17^3 = 4\ 9\ 1\ 3$	$4+9+1+3=17$
$197^3 = 7\ 64\ 53\ 73$	$7+64+53+73=197$
$1997^3 = 7\ 964\ 053\ 973$	$7+964+053+973=1997$
$19997^3 = 7\ 9964\ 0053\ 9973$	$7+9964+0053+9973=19997$
$199997^3 = 7\ 99964\ 00053\ 99973$	$7+99964+00053+99973=199997$
$1999997^3 = 7\ 999964\ 000053\ 999973$	$7+999964+000053+999973=1999997$

Same also for 18, 198, 1998, etc.

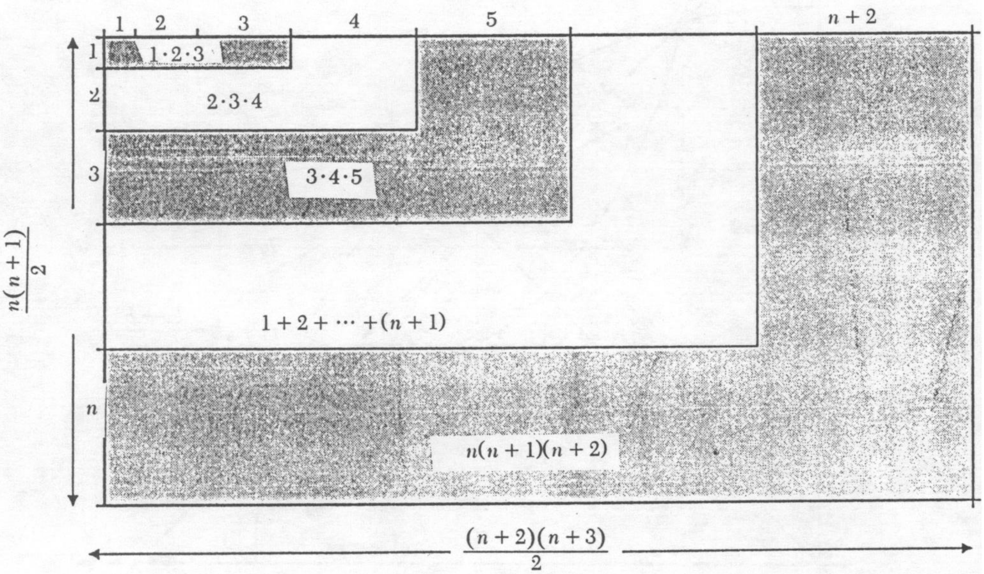
—I. A. Sakmar
Physics Department
University of South Florida
Tampa, FL 33620

Proof without Words:

$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$



$$k\frac{(k+1)(k+2)}{2} + (k+2)\frac{k(k+1)}{2} = k(k+1)(k+2).$$



$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + n(n+1)(n+2)$$

$$= \frac{n(n+1)}{2} x \frac{(n+2)(n+3)}{2} = \frac{n(n+1)(n+2)(n+3)}{4}$$

—James O. Chilaka
Long Island University
Brookville, NY 11548

Proof without Words: Five Means—and Their Means

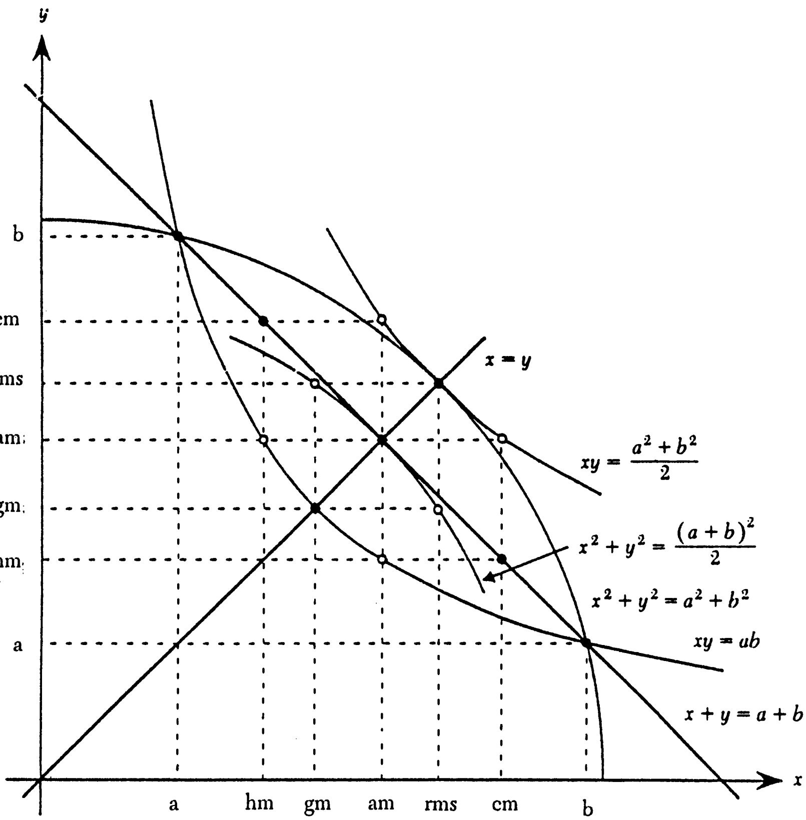
Arithmetic: $\mathrm{am} = \mathrm{AM}(a, b) = \dfrac{a+b}{2}$

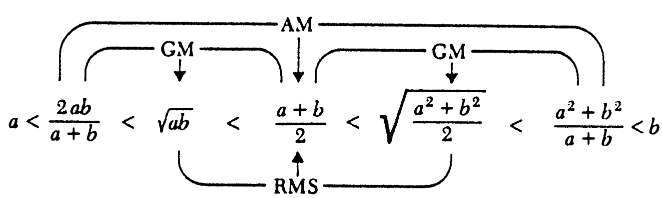
Contraharmonic: $\mathrm{cm} = \mathrm{CM}(a, b) = \dfrac{a^2+b^2}{a+b}$

Geometric: $\mathrm{gm} = \mathrm{GM}(a, b) = \sqrt{ab}$

Harmonic: $\mathrm{hm} = \mathrm{HM}(a, b) = \dfrac{2\,ab}{a+b}$

Root mean square: $\mathrm{rms} = \mathrm{RMS}(a, b) = \sqrt{\dfrac{a^2+b^2}{2}}$





I. $0<a<b \Rightarrow a<\frac{2ab}{a+b}<\sqrt{ab}<\frac{a+b}{2}<\sqrt{\frac{a^2+b^2}{2}}<\frac{a^2+b^2}{a+b}<b.$

II. $\text{hm}+\text{cm}=a+b \Rightarrow \text{AM(hm, cm)}=\text{am}$

III. $\text{hm}\cdot\text{am}=a\cdot b \Rightarrow \text{GM(hm, am)}=\text{gm}$

IV. $\text{am}\cdot\text{cm}=\frac{a^2+b^2}{2} \Rightarrow \text{GM(am, cm)}=\text{rms}$

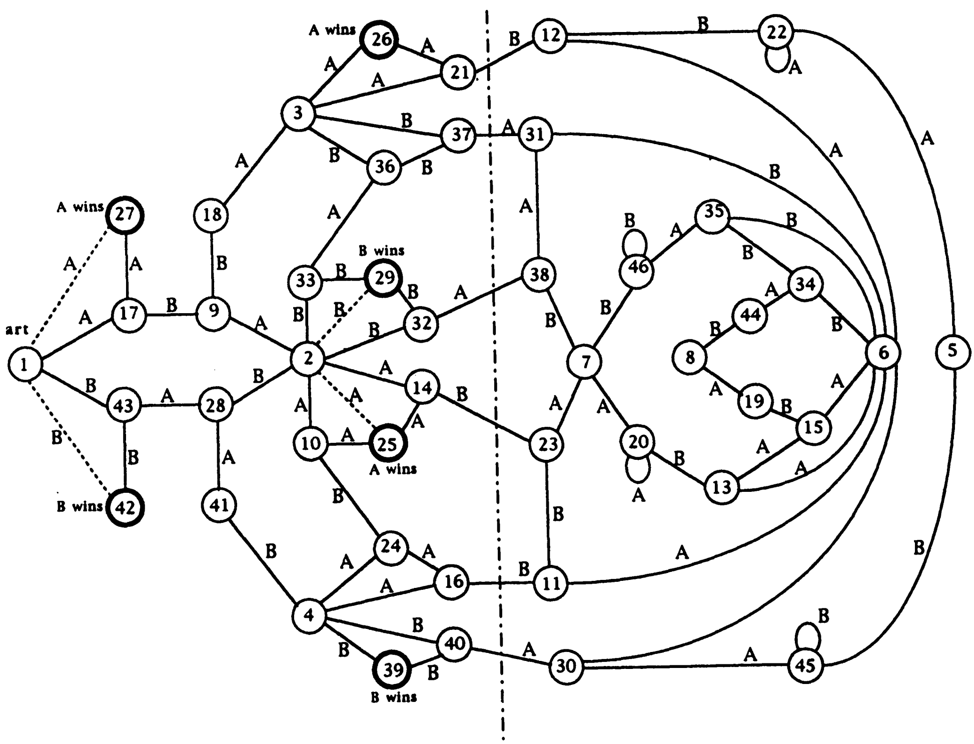
V. $\text{gm}^2+\text{rms}^2=\frac{(a+b)}{2} \Rightarrow \text{RMS(gm, rms)}=\text{am}$

—ROGER B. NELSEN
LEWIS AND CLARK COLLEGE
PORTLAND, OR 97219

Corrected Figure for Position Graphs for Pong Hau K'i and Mu Torere

In *Position Graphs for Pong Hau K'i and Mu Torere*, by Philip D. Straffin, Jr., in the December 1995 issue of this MAGAZINE, the edge labels on the graph in FIGURE 5, page 384, were inadvertently omitted. The correct figure appears below.

We regret the error. - Ed.



$$\text{I.}\quad 0<a<b \;\Rightarrow\; a<\frac{2ab}{a+b}<\sqrt{ab}<\frac{a+b}{2}<\sqrt{\frac{a^2+b^2}{2}}<\frac{a^2+b^2}{a+b}<b.$$

$$\text{II.}\quad \mathrm{hm}+\mathrm{cm}=a+b \;\Rightarrow\; \mathrm{AM}(\mathrm{hm},\mathrm{cm})=\mathrm{am}$$

$$\text{III.}\quad \mathrm{hm}\cdot\mathrm{am}=a\cdot b \;\Rightarrow\; \mathrm{GM}(\mathrm{hm},\mathrm{am})=\mathrm{gm}$$

$$\text{IV.}\quad \mathrm{am}\cdot\mathrm{cm}=\frac{a^2+b^2}{2} \;\Rightarrow\; \mathrm{GM}(\mathrm{am},\mathrm{cm})=\mathrm{rms}$$

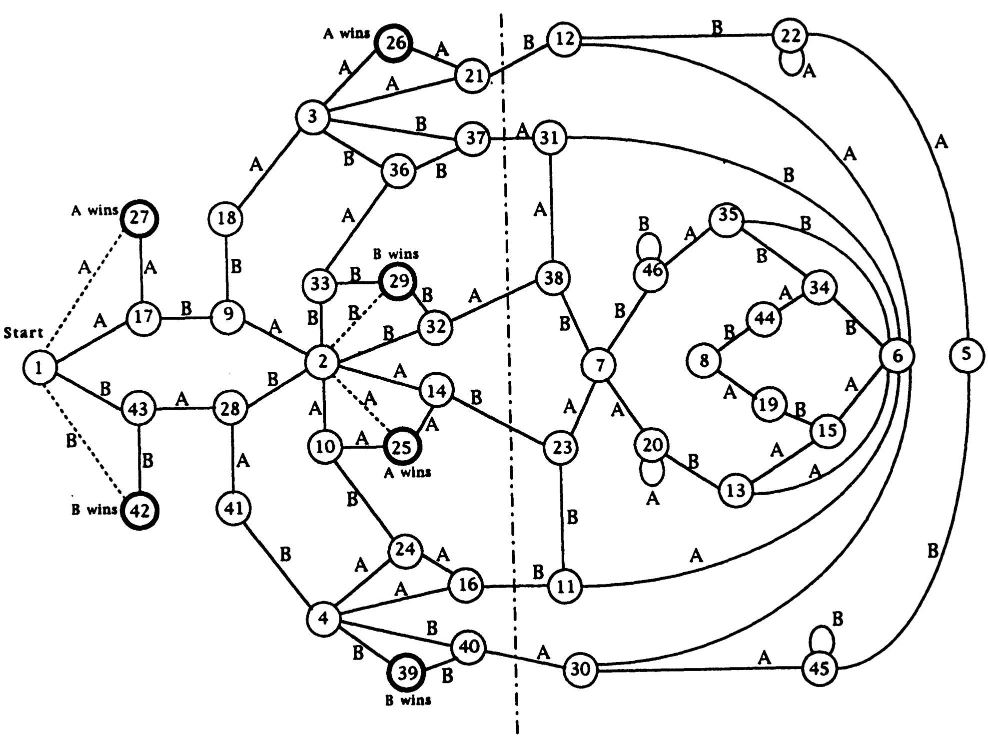
$$\text{V.}\quad \mathrm{gm}^2+\mathrm{rms}^2=\frac{(a+b)}{2} \;\Rightarrow\; \mathrm{RMS}(\mathrm{gm},\mathrm{rms})=\mathrm{am}$$

—ROGER B. NELSEN
LEWIS AND CLARK COLLEGE
PORTLAND, OR 97219

Corrected Figure for Position Graphs for Pong Hau K'i and Mu Torere

In *Position Graphs for Pong Hau K'i and Mu Torere*, by Philip D. Straffin, Jr., in the December 1995 issue of this MAGAZINE, the edge labels on the graph in FIGURE 5, page 384, were inadvertently omitted. The correct figure appears below.

We regret the error. - Ed.



PROBLEMS

GEORGE T. GILBERT, *editor*
Texas Christian University

ZE-LI DOU, KEN RICHARDSON, and SUSAN G. STAPLES, *assistant editors*
Texas Christian University

Proposals

To be considered for publication, solutions should be received by July 1, 1996

1489. *Proposed by David Callan, University of Wisconsin, Madison, Wisconsin.*

For what integers k is

$$\frac{m!n!\operatorname{lcm}\{1,2,\ldots,m+n+k\}}{(m+n+k)!}$$

an integer for all nonnegative integers m and n such that $m+n+k>0$ (lcm denoting the least common multiple)?

1490. *Proposed by Evgenii S. Freidkin, Rutgers—The State University of New Jersey, Piscataway, New Jersey.*

A mouse with maximum speed v_m sits at the center of a regular pentagon. At each vertex of the pentagon is a cat with maximum speed v_c. If the cats must remain on the boundary of the pentagon, find a necessary and sufficient condition which guarantees the mouse can escape from the pentagon. (Assume that the animals are points and that changes in velocity may be instantaneous.)

1491. *Proposed by Wu Wei Chao, He Nan Normal University, Xin Xiang City, He Nan Province, China.*

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

(i) $f(x+f(y)+yf(x)) = y+f(x)+xf(y)$ for all x, y, in $\mathbb{R}$;

(ii) $\{f(x)/x\colon x \in \mathbb{R},\ x \neq 0\}$ is a finite set.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution.

Solutions should be written in a style appropriate to this MAGAZINE. *Each solution should begin on a separate sheet containing the solver's name and full address.*

Solutions and new proposals should be mailed to George T. Gilbert, Problems Editor, Department of Mathematics, Box 32903, Texas Christian University, Fort Worth, TX 76129, or mailed electronically (ideally as a LaTeX file) to g.gilbert@tcu.edu. Readers who use e-mail should also provide an e-mail address.

1492. *Proposed by Michael Golomb, Purdue University, West Lafayette, Indiana.*

Derive the Laplace transform of the function $(2\sin x - \sin 2x)/x^2$; i.e. for $\operatorname{Re} s > 0$ evaluate the integral

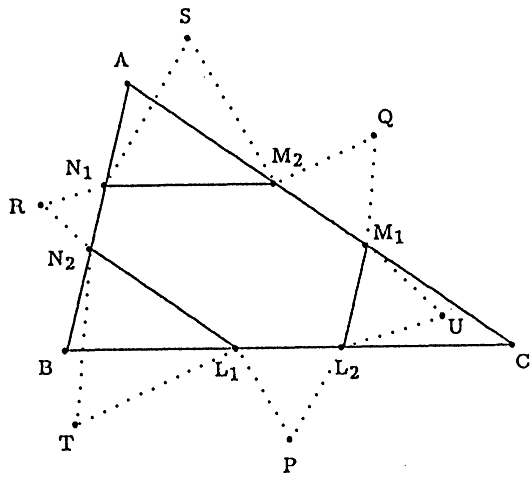
$$\int_0^\infty e^{-sx}\frac{2\sin x - \sin 2x}{x^2}\,dx.$$

1493. *Proposed by Jiro Fukuta, Gifu-ken, Japan.*

In ΔABC, let L_1 and L_2, M_1 and M_2, N_1 and N_2 be distinct points on the sides BC, CA, AB, respectively, such that

$$\frac{BL_1}{L_1C} = \frac{CL_2}{L_2B} = \frac{CM_1}{M_1A} = \frac{AM_2}{M_2C} = \frac{AN_1}{N_1B} = \frac{BN_2}{N_2A} < 1.$$

Let PL_1L_2, QM_1M_2, RN_1N_2, SM_2N_1, TN_2L_1, and UL_2M_1 be the equilateral triangles built outwards on the sides of the hexagon $L_1L_2M_1M_2N_1N_2$.



(i) Prove that the segments PS, QT, and RU have equal lengths, and the lines PS, QT, and RU intersect at 120° angles and are concurrent.

(ii) If G_1, G_2, G_3, G_4, G_5, and G_6 are the centroids of triangles QSR, SRT, RTP, TPU, PUQ, and UQS, respectively, prove that $G_1G_2G_3G_4G_5G_6$ is a regular hexagon whose centroid coincides with that of ΔABC.

Quickies

Answers to the Quickies are on page 73.

Q844. *Proposed by Erwin Just (Emeritus) and Norman Schaumberger (Emeritus), Bronx Community College, New York, New York.*

Let (a, b, c) be a Pythagorean triple.

(i) Prove that there exists an integer t such that at, bt, and ct are the lengths of the sides of a right triangle in the plane with all three vertices at lattice points and with neither leg parallel to an axis.
(ii) If (a, b, c) is primitive, is it possible that $t = 1$?

Q845. *Proposed by William P. Wardlaw, U.S. Naval Academy, Annapolis, Maryland.*

Let A be a square matrix, over a field, for which there exist positive integers m and n such that $A^m = I \neq A^n$. Show that $\det(I + A +^2 + \cdots + A^{m-1}) = 0$.

Q846. *Proposed by John P. Hoyt, Lancaster, Pennsylvania.*

Show that $\cos(\pi/7)\cdot\cos(2\pi/7)\cdot\cos(3\pi/7) = 1/8$.

Solutions

Rational Solutions to an Exponential Equation **February 1995**

1464. *Proposed by Bill Correll, Jr., student, Denison University, Granville, Ohio.*

Find all positive rational numbers $r \neq 1$ such that $r^{1/(r-1)}$ is rational.

Solution by Achilleas Sinefakopoulos, student, University of Athens, Greece.

The set of solutions is $\{1 + 1/m \colon m \in \mathbb{Z},\ m \neq 0, -1\}$. We first tackle the case $r > 1$. Let $r = 1 + n/m$, where n and m are relatively prime positive integers. If

$$r^{1/(r-1)} = \left(\frac{m+n}{m}\right)^{m/n}$$

is rational, then there exist relatively prime positive integers a and b such that $m + n = a^n$ and $m = b^n$. If $n > 1$, then by the mean value theorem

$$n = a^n - b^n > nb^{n-1}(a-b) \geq n,$$

an obvious contradiction. This forces $n = 1$, and $r = 1 + 1/m$, for $m = 1, 2, 3, \ldots$, which is easily seen to be a solution. The case $0 < r < 1$ is handled similarly, yielding solutions of the form $r = 1 - 1/m$, for $m = 2, 3, \ldots$.

Comments. Peter Lindstrom notes that $s = r^{1/(r-1)}$ implies $s^r = rs$. In this form, the equation is Problem 127 in *The Two-Year College Mathematics Journal* 9 (1978), 297.
Juan-Bosco Romero Mařquez refers us to problem 1688 in *Crux Mathematicorum* 18 (1992), 279–280, where the equation appears in the form $r \log s = \log(rs)$. Robin Chapman observes that substituting $y = rx$ into the well-known equation $x^y = y^x$ yields $x = r^{1/(r-1)}$.

Also solved by Brian D. Beasley, Kenneth L. Bernstein, David M. Bloom, Stan Byrd, Robin Chapman (U.K.), Shawn Godin (Canada), Gerald A. Heuer, Dragan Janković, D. Kipp Johnson, Raymond Lai, Kee-Wai Lau (Hong Kong), Peter A. Lindstrom, O. P. Lossers (The Netherlands), Juan-Bosco Romero

Marquez (Spain), Mark McKinzie, Stephen Noltie, Allan Pedersen (Denmark), John F. Putz, Fary Sami and Reza Akhlaghi, David Stone, Monte J. Zerger, and the proposer. There were three incomplete and two incorrect solutions.

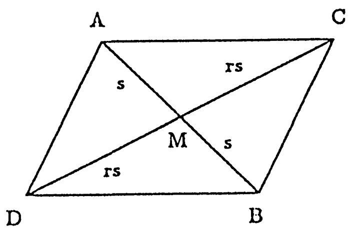
Two-Colorings of the Plane **February 1995**

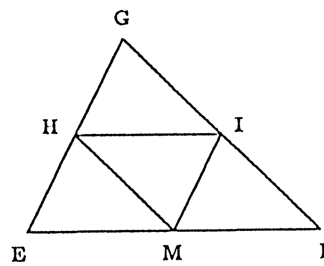
1465. *Proposed by Stephen W. Knox, University of Illinois, Urbana, Illinois.*

It is a well-known theorem that given any coloring of the plane by two colors, there exists an equilateral triangle with monochromatic vertices. As a generalization, show that given any two-coloring of the plane and any triangle T, there exists a triangle similar to T with monochromatic vertices.

I. *Solution by Roger B. Eggleton, Illinois State University, Normal, Illinois.*

Denote the vertices of T, which may be degenerate, by P, Q, and R. Let r denote the ratio of the length of the median from R to the length of PQ. For any positive s we claim there exists a monochromatic triangle similar to T with ratio of similitude in the set $\{s, 2s, rs, 2rs\}$. To see this, we may assume $PQ = 1$. Choosing appropriate vertices from any equilateral triangle of side $2s$, let $\{A, B\}$ be a monochromatic pair with $AB = 2s$ and M the midpoint of AB. Consider a parallelogram $ACBD$ with $\Delta ABC \sim \Delta BAD \sim \Delta PQR$.





If C or D is the same color as A and B, then ΔABC or ΔBAD is a monochromatic triangle similar to T, with ratio of similitude $2s$. Otherwise, either $\{A, B, M\}$ or $\{C, D, M\}$ is a monochromatic triple. Thus, there exists a monochromatic triple $\{E, F, M\}$ with $EF = 2s$ or $2rs$ and M the midpoint of EF. Choose G such that $\Delta EFG \sim \Delta PQR$ and let H and I be the midpoints of EG and FG, respectively. If, G, H, or I has the same color as $\{E, F, M\}$, we have a monochromatic triangle similar to T with ratio of similitude in the set $\{s, 2s, rs, 2rs\}$. If not, ΔGHI is monochromatic and similar to T, with ratio of similitude s or rs.

II. *Solution by Thomas Leong, student, City College of CUNY, New York, New York.*

We show more generally that the result holds for any N-coloring of the plane. Denote the vertices of a triangle similar to T by the vectors $\mathbf{0}$, $\mathbf{v}_1$, and $\mathbf{v}_2$. Let the "triangular lattice" $\Gamma_n(\mathbf{u}, \mathbf{v}, \mathbf{w})$ be defined by

$$\Gamma_n(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \{\mathbf{u} + i\mathbf{v} + j\mathbf{w}: i, j \in \mathbb{Z}, 0 \le i, 0 \le j, i + j \le n\}.$$

We apply van der Waerden's theorem:

For all positive integers k and r, there exists an integer $W(k, r)$ so that, if the set of integers $\{1, 2, \dots, W(k, r)\}$ is partitioned into r classes, then at least one class contains a k-term arithmetic progression.

Define integers W_N recursively by $W_1 = 1, W_{N+1} = W(W_N + 2, N+1) - 1$. We shall show by induction on N that if $\Gamma_{W_N}(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)$ is N-colored, then it contains a monochromatic triangle similar to T. The case $N = 1$ is clear. Assume the claim for N and consider an $N+1$-coloring of the plane. Applying van der Waerden's theorem to the set $\{1, 2, \dots, W_{N+1} + 1\}$, it follows that the "base row" of $\Gamma_{W_{N+1}}(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)$ contains a monochromatic set of the form

$$B = \{(a + kd): k = 0, 1, \dots, W_N + 1\}.$$

The sublattice with B as its base row is the union of B and

$$\Gamma_{W_N} = \Gamma_{W_N}(a\mathbf{v}_1 + d\mathbf{v}_2, d\mathbf{v}_1, d\mathbf{v}_2).$$

If any member of Γ_{W_N} has the same color as B, we are done. If not, we apply the inductive hypothesis to Γ_{W_N} which is simply a translation and dilation of $\Gamma_{W_N}(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)$.

Comments. Eggleton also proves by reasoning similar to that of the first solution that there is a monochromatic triangle similar to T with ratio of similitude in the set $\{s, 2s, 3s, 4s, 6s\}$.

Also solved by Kenneth L. Bernstein, Robin Chapman (U.K.), Jerrold W. Grossman, Nick Lord (England), O. P. Lossers (The Netherlands), Eric Fabian Hernandez Martinez (Mexico), Achilleas Sinefakopoulos (student, Greece), Saul Stahl, and the proposer.

Existence of Two Subsets with Equal Sums **February 1995**

1466. *Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, New York.*

Let m and n be positive integers. If $x_1, \dots, x_m$ are positive integers whose average is less than $n+1$ and if $y_1, \dots, y_n$ are positive integers whose average is less than $m+1$, prove that some sum of one or more x's equals some sum of one or more y's. (This is a strengthening of Putnam problem A-4, 1993; see this MAGAZINE, April 1994, 156–157.)

Solution by Robin Chapman, University of Exeter, Exeter, U.K.

For $0 \le j \le m$ and $0 \le k \le n$ define

$$s_j = \sum_{i=1}^{j} x_i \text{ and } t_k = \sum_{i=1}^{k} y_i.$$

If $s_m = t_n$ we are done, so assume by symmetry that $s_m > t_n$. Define the set $A = \{(j, k): 0 \le j < m, 0 \le k \le n\}$. This set has $m(n+1) > s_m$ elements, and so, by the pigeonhole principle, there exist distinct elements (j, k) and (j', k') of A with

$$s_j + t_k \equiv s_{j'} + t_{k'} \pmod{s_m},$$

or, equivalently,

$$s_j - s_{j'} \equiv t_{k'} - t_k \pmod{s_m}.$$

We may assume that $k' \ge k$. Since $|s_j - s_j'| < s_m$ and $0 \le t_{k'} - t_k \le t_n < s_m$, we see that $t_{k'} - t_k > 0$. Either $s_j - s_{j'} = t_{k'} - t_k$ or $s_j - s_{j'} = t_{k'} - t_k - s_m$. In the former case,

$$\sum_{i=j'+1}^{j} x_i = \sum_{i=k+1}^{k'} y_i,$$

and in the latter,

$$\sum_{i=1}^{j} x_i + \sum_{i=j'+1}^{m} x_i = \sum_{i=k+1}^{k'} y_i.$$

Also solved by S. F. Barger and the proposer.

Vertex Sums of a Regular Simplex **February 1995**

1467. *Proposed by John A. Baker, University of Waterloo, Waterloo, Ontario, Canada.*

Suppose that $v_0, v_1, \dots, v_n$ are the vertices of a regular simplex, S, in $\mathbb{R}^n$ centered at the origin. Let

$$v_i = (v_{i1}, v_{i2}, \dots, v_{in}) \qquad \text{for } 0 \le i \le n.$$

Prove that, for some $c > 0$,

$$\sum_{i=0}^{n} v_{ij}^2 = c \qquad \text{for all } j = 1, 2, \dots, n.$$

Solution by Sung Eun Koh, Konkuk University, Seoul, Korea.

Write $\mathbf{v}_i$ for the vector from the origin to v_i. Let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary vectors in $\mathbb{R}^n$. A solution follows from the more general identity:

$$\sum_{i=0}^{n} (\mathbf{v}_i \cdot \mathbf{x})(\mathbf{v}_i \cdot \mathbf{y}) = \left(1 + \frac{1}{n}\right)\|\mathbf{v}_0\|^2 (\mathbf{x} \cdot \mathbf{y}).$$

Since the regular simplex S is centered at the origin,

$$\mathbf{v}_0 + \cdots + \mathbf{v}_n = \mathbf{0}.$$

It suffices to prove the result assuming $\|\mathbf{v}_i\| = 1$. In this case $\mathbf{v}_i \cdot \mathbf{v}_j = -1/n$ for $i \ne j$. Multiplying by a small scalar, we may also assume that $\mathbf{x}$ and $\mathbf{y}$ are inside S. Then there exist positive λ_i, μ_i, $i = 0, 1, \dots, n$, such that

$$\mathbf{x} = \lambda_0 \mathbf{v}_0 + \cdots + \lambda_n \mathbf{v}_n,\ \lambda_0 + \cdots + \lambda_n = 1,$$
$$\mathbf{y} = \mu_0 \mathbf{v}_0 + \cdots + \mu_n \mathbf{v}_n,\ \mu_0 + \cdots + \mu_n = 1.$$

Since

$$\mathbf{v}_i \cdot \mathbf{x} = \mathbf{v}_i \cdot (\lambda_0 \mathbf{v}_0 + \cdots + \lambda_n \mathbf{v}_n) = \left(1 + \frac{1}{n}\right)\lambda_i - \frac{1}{n},$$

$$\mathbf{v}_i \cdot \mathbf{y} = \mathbf{v}_i \cdot (\mu_0 \mathbf{v}_0 + \cdots + \mu_n \mathbf{v}_n) = \left(1 + \frac{1}{n}\right)\mu_i - \frac{1}{n},$$

we see that

$$\mathbf{x}\cdot\mathbf{y}=\left(1+\frac{1}{n}\right)\sum_{i=0}^{n}\lambda_i\mu_i-\frac{1}{n} \tag{1}$$

and that

$$\begin{aligned}\sum_{i=0}^{n}(\mathbf{v}_i\cdot\mathbf{x})(\mathbf{v}_i\cdot\mathbf{y})&=\sum_{i=0}^{n}\left(\left(1+\frac{1}{n}\right)\lambda_i-\frac{1}{n}\right)\left(\left(1+\frac{1}{n}\right)\mu_i-\frac{1}{n}\right)\\&=\left(1+\frac{1}{n}\right)^2\sum_{i=0}^{n}\lambda_i\mu_i-\frac{1}{n}\left(1+\frac{1}{n}\right).\end{aligned} \tag{2}$$

Comparing (1) and (2) we obtain

$$\sum_{i=0}^{n}(\mathbf{v}_i\cdot\mathbf{x})(\mathbf{v}_i\cdot\mathbf{y})=\left(1+\frac{1}{n}\right)(\mathbf{x}\cdot\mathbf{y})$$

when $\|\mathbf{v}_i\|=1$, and the claim follows.

Comment. Nick Lord points out the close relationship between this problem and problem 1916 in *Crux Mathematicorum* 20 (1994), 48, which asserts that for a unit vector $\mathbf{x}$,

$$\sum_{i=0}^{n}\|\mathbf{v}_i-\mathbf{x}\|^4=\frac{4(n+1)^2}{n}.$$

Also solved by Robin Chapman (U.K.), Robert L. Doucette, Nick Lord (England), Robert Patenaude, Van Vu Ha, and the proposer.

Solutions of a Matrix Equation **February 1995**

1468. *Proposed by G. Trenkler, University of Dortmund, Dortmund, Germany.*

Let A be a square matrix with real entries satisfying $A^2=A^T$.

(i) Find its Moore-Penrose inverse A^+ in terms of A.
(ii) Assume A is a 2×2 matrix. Find all solutions to $A^2=A^T$ that are not symmetric.

(i) *Solution by Stan Byrd and Ronald L. Smith, University of Tennessee at Chattanooga, Chattanooga, Tennessee.*

We will solve (i) in the more general case in which A is complex and $A^k=A^*$, the conjugate transpose of A, for some integer $k\geq 2$. Consequently, $A=(A^k)^*=(A^*)^k$. Then $A^{k^2}=(A^k)^k=(A^*)^k=A$ and $A^{k+1}=AA^k=(A^*)^{k+1}$. For $X=A^{k^2-2}$, it follows that

(a) $AXA=A^{k^2}=A$,
(b) $XAX=A^{2k^2-3}=A^{k^2-2}=X$,
(c) $(AX)^*=(A^{k^2-1})^*=((A^{k+1})^{k-1})^*=(((A^*)^{k+1})^{k-1})^*=A^{k^2-1}=AX$,

and similarly,

(d) $(XA)^*=XA$,

which proves $A^+=A^{k^2-2}$.

(ii) *Solution by Ahmad Muchlis, Institut Teknologi Bandung, Bandung, Indonesia.*

Let A be a 2×2 nonsymmetric, real matrix satisfying $A^2 = A^T$. The condition $A^2 = A^T$ implies that $A^4 = A$. Then the minimal polynomial of A, $m(\lambda)$ is a factor of $\lambda^4 - \lambda = \lambda(\lambda - 1)(\lambda^2 + \lambda + 1)$. The nonsymmetry of A forces $m(\lambda) = \lambda^2 + \lambda + 1$, hence A is nonsingular. It follows from (i) that $A^{-1} = A^+ = A^T$, i.e., A is orthogonal. Therefore A is a rotation matrix, i.e.,

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \neq 0$. From $A^2 = A^T = A^{-1}$, we obtain $2\theta = -\theta + 2k\pi$, k an integer. For $0 < \theta < 2\pi$, we have $\theta = 2\pi/3$ or $\theta = 4\pi/3$. We conclude that there are only two 2×2 nonsymmetric, real matrices A satisfying $A^2 = A^T$, namely

$$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \text{ and } \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}.$$

Also solved by Michael H. Andreoli, S. F. Barger, Glenn A. Bookhout, Robin Chapman (U.K.), Adam Coffman, Robert L. Doucette, Tim Flood, Edwin T. Hoefer, Michael K. Kinyon, Gary Miller, Nicholas C. Singer, Trinity University Problem Group, David Zhu, and the proposer.

Answers

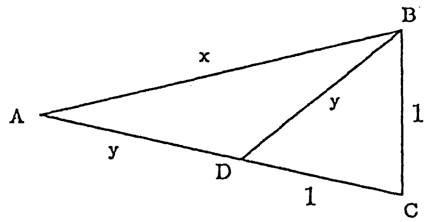
Solutions to the Quickies on page 67.

A844. (i) Let t be a positive integer such that $t^2 = x^2 + y^2$ for positive integers x and y. Defining $A = (by, bx)$, $B(ax, -ay)$, and $C = (0,0)$, we compute that $BC = at$, $AC = bt$, and $AB = ct$. Thus ΔABC is a solution to (i).

(ii) It is impossible to have $t = 1$ for a primitive Pythagorean triple. Assume that there exist lattice points A, B, and C, as required, satisfying $BC = a$, $AC = b$, and $AB = c$, where (a, b, c) is a primitive Pythagorean triple. Without loss of generality, we may assume that $C = (0,0)$, so that $A = (kx, ky)$ and $B = (my, -mx)$ for nonzero integers k, m, x, and y. Then $a^2 = m^2(x^2 + y^2)$ and $b^2 = k^2(x^2 + y^2)$. Because a and b are relatively prime, $x^2 + y^2 = 1$, a contradiction.

A845. Since $0 = A^m - I = (A - I)(I + A + A^2 + \cdots + A^{m-1})$ and $A - I \neq 0$, it follows that $I + A + A^2 + \cdots + A^{m-1}$ is singular, so $\det(I + A - A^2 + \cdots + A^{m-1}) = 0$.

A846. Let ABC be an isosceles triangle with $AB = AC = x$, $BC = 1$, $\angle A = \pi/7$, $\angle B = \angle C = 3\pi/7$. Choose D on AC so that $\angle ABD = \pi/7$. Hence ΔABD and ΔBCD are isosceles. Let $y = AD = BD$. We have $\cos(\pi/7) = \cos\angle A = x/(2y)$, $\cos(2\pi/7) = \cos\angle CBD = y/2$, and $\cos(3\pi/7) = \cos\angle C = 1/(2x)$, and the result follows.



Comments

S1452, Concurrent Lines in a Triangle, June 1995. Peter Yff writes that "the locus of the point of concurrence is a conic known as Kiepert's hyperbola. This is a rectangular hyperbola passing through A, B, C, the centroid, the orthocenter, the Spieker center, the isogonic centers, and the Napoleon points." He refers to page 223 of R. A. Johnson's *Advanced Euclidean Geometry*.

S1454, Names Drawn From a Hat, October 1995. Hugh McGuire was inadvertently omitted from the list of those who solved the problem.

S1455, Equilateral Triangles in a Hexagon, October 1995. Michael Vowe (Switzerland) was inadvertently omitted from the list of those who solved the problem.

REVIEWS

PAUL J. CAMPBELL, *editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Cipra, Barry, *What's Happening in the Mathematical Sciences: Vol. 3, 1995–1996*, AMS, 1996; vi + 111 pp, $12 (P). ISBN 0-8218-0355-7.

As in earlier volumes in this series, Cipra reports on ten recent developments and discoveries in mathematics, in a lively nontechnical style. This volume's topics are the finishing off of Fermat's Last Theorem; Witten and Seiberg's discoveries in quantum field theory and four-dimensional geometry; computer science's realization that DNA can be a computational medium; Nicely's revelation of the flaw in the Pentium chip; new developments in control theory, computational fluid dynamics, and cellular automata; the attack to simplify the proof of the classification theorem for simple groups; the factoring of RSA–129; and applications of calculus in finance. You and your students need to have this book!

Special Section on Mathematics Applied to Games. *The UMAP Journal* 16 (1) (1995) 9–36, 71–77; 16 (2) (1995) 93–184; 16 (4) (1995) 341–88. Floyd, Jeffrey K., A discrete analysis of "Final Jeopardy," *Mathematics Teacher* 87 (5) (May 1994) 328–331. Woodward, Ernest, and Marilyn Woodward, Expected value and the Wheel of Fortune game, *Mathematics Teacher* 87 (1) (January 1994) 13–17.

Games have always inspired mathematicians and mathematics. Recently there has been a new renascence of combinatorial game theory: The Mathematical Sciences Research Institute (MSRI) held a workshop on the topic, and John Horton Conway spoke there and at the 1994 International Congress of Mathematicians about the Hawaiian game of Konane and about mancala-type games. The Special Section in *The UMAP Journal* is spread over three issues; it treats both combinatorial games and games of chance. The first issue offers the optimal strategy for the German dice game Klappenspiel, a surprise π in the combinatorics of mancala-type games (with proof of a 40-year-old conjecture of Erdös and Jabotinsky), and opportunities for student research. The second issues features a mathematical analysis plus anthropological details of Konane, optimal strategies for the television game The Price Is Right, the probability of winning at Frustration solitaire, and an instructional module on the combinatorics of pinochle hands. The third issue contains a computer-aided analysis of a mancala solitaire plus anthropological details, and an instructional module on Nim. The cited papers in the *Mathematics Teacher* apply probability to popular television games: Floyd gives an optimal strategy for the leader entering Final Jeopardy, and Woodward and Woodward determine when a player should to continue to spin the Wheel of Fortune if all players know the puzzle's solution. (Note: I am the editor of *The UMAP Journal* and co-author of one of the articles in the Special Section.)

Amir, Amihood, and members of the SIGACT Long Range Planning Committee, Contributions of theoretical computer science, `http://www.cs.umd.edu/~smith/cont/cont.html`, 11 December 1995.

This too-brief document attempts to "recognize the recent contributions of theoretical computer science both to the practice of computing and to other endeavors of society." It mentions new algorithms for cryptography, computational geometry, ATM (not automatic teller machine but asynchronous transfer mode) networks, randomization in routing messages in a parallel machine, efficient scheduling of multi-threaded computations, VLSI design, and computational learning theory (handwriting recognition, speech recognition, DNA sequence modeling). Cited as contributions to other disciplines are "sparse" dynamic programming (for DNA sequence comparison), domain theory (developed as a foundation for the semantics of programming languages, now applied in fractals, neural nets, and the Ising model in statistical physics), an algorithm for convex hulls (used to determine needed support structures in layered manufacturing), and network flow techniques (used to optimize telescope settings). This potentially inspiring working document would be much more valuable if expanded to five or ten times the length (it's now three pages), to include definitions, further discussion, and citations to references and specific applications.

Wilmott, Paul, Sam Howson, and Jeff Dewynne, *The Mathematics of Financial Derivatives: A Student Introduction*, Cambridge U Pr, 1995; xiii + 317 pp, $49.95, $24.95 (P). ISBN 0-521-49699-3, 0-521-49789-2.

With the growth of all kinds of financial derivatives, there are substantial job opportunities for mathematics majors in the finance industry. This book, which is suitable for a topics seminar, would help them on their way, though they may need supplementary background to become comfortable with probability density functions, partial differential equation initial-value problems, and finite-difference and SOR methods for obtaining numerical solutions. Each chapter has exercises, with hints at the back of the book.

Robinson, Philip, Evangelista Torricelli, *Mathematical Gazette* 79 (1995) 37–47.

Torricelli (1608–1647) is famous as the inventor of the barometer. Briefly an assistant to Galileo before succeeding him at the University of Pisa, Torricelli put Galileo's work on projectiles on a firm mathematical basis. It was Torricelli who first proved that the maximum range in a vacuum is achieved for a launch angle of 45°, and he prepared gunnery tables and a ranging instrument that were fundamental to ballistics for a century. This article reproduces figures, arguments, and tables from Torricelli's work.

Ballew, Hunter, Sherlock Holmes: Master problem solver, *Mathematics Teacher* 87 (8) (November 1994) 596–601.

Author Hunter examines the adventures of detective Sherlock Holmes to exhibit for students the connections between Holmes' methods and mathematical problem solving. Apart from deduction, Holmes emphasized careful observation and examination of details, an open mind, the importance of gathering data, questioning the obvious, learning from experience, and indirect proof ("[W]hen you have excluded the impossible, whatever remains, however improbable, must be the truth.") Hunter quotes from specific adventures to illustrate Homes's use of these problem-solving techniques. In addition, two adventures have Holmes solving mathematical problems (estimating the speed of a train from the time to pass consecutive telegraph poles, and using similar triangles to determine the height of a tree that had been cut down).

Stewart, Ian, The anthropomurphic [sic] principle, *Scientific American* 273 (6) (December 1995) 104–105. Matthews, R.A.J., Tumbling toast, Murphy's law and the fundamental constants. *European Journal of Physics* 16 (4) (1995) 172–176.

Buttered toast falling off a table seems always to land buttered side down. Is this perversity just Murphy's Law at work? ("If anything can go wrong, it will.") Mathematical journalist Matthews's analysis of the "murphodynamics" of falling toast reveals instead a "murphic resonance" among the size of a piece of toast, the standard height of a table, and the earth's gravity. The toast must rotate at least 180°; but to land butter side up, it would have to rotate at least 360° (so say Matthews and Stewart, but a trifle more than 270° would seem to be enough). That can happen, but only if the overhang of the toast when it begins to pivot and fall is sufficiently large. Further calculation, of the limiting height of bipedal organisms, shows that the result does not really depend on the parameter values: "Any universe that contains creatures remotely like us will necessarily inflict Murphy's Law on its inhabitants—at least if they eat toast and sit at tables."

Denley, Chris, and Chris Pritchard, The golf ball aerodynamics of Peter Guthrie Tait, *Mathematical Gazette* (1995) 298–313.

Peter Guthrie Tait was a professor of physics at the University of Edinburgh in the last half of the nineteenth century. He was also an avid golfer (as many as five rounds a day!), and that hobby inspired him to model the flight of a golf ball. This article relates several of his models and compares them with computer simulations. Tait died before the advent of dimpled golf balls, whose greater range depends on reduced drag because of turbulence.

Peterson, Ivars, Crinkled doughnuts: Math in the folds of a polyhedral crown, *Science News* 148 (23 & 30 December 1995) 432–433.

William T. Webber, a graduate student at the University of Washington, discovered how to fold a toroidal polyhedron out of a piece of paper ruled into identical triangular faces. Variations in the triangle produce variations in the toroidal "crown"; curiously, he has not yet discovered such a crown that derives from an equilateral triangle.

Barbeau, Ed, *After Math: Puzzles and Brainteasers*, Wall & Emerson, 1995; x + 198 pp, $14.95 (P). ISBN 0-921332-42-4. Barbeau, Edward J., Murray S. Klamkin, and William O.J. Moser, *Five Hundred Mathematical Challenges*, MAA, 1995; xi + 227 pp, $29.50 (P). ISBN 0-88385-519-4. Cofman, Judita, *Numbers and Shapes Revisited: More Problems for Young Mathematicians*, Oxford U Pr, 1995; xi + 308 pp (P). ISBN 0-19-853460-4.

Many of the problems in Barbeau's own book are ones that he originally contributed to the University of Toronto *Alumni Magazine* and a supplementary newsletter. The problems do not require mathematics beyond high school; solutions, comments, and extensions are included. Most of the problems involve geometry, but here is a brief sample that does not require a figure: "Find all positive integers whose square is equal to the fifth power of the sum of its digits." In competition with himself, Barbeau is also co-editor of a volume of 500 "problems in pre-calculus mathematics," aimed at an audience of students who find pleasure in wrestling with and overcoming problems that are challenging and though-provoking. Brief sample: "Show that every simple polyhedron has at least two faces with the same number of edges." The third collection of problems, by Cofman, is aimed at "advanced secondary school pupils, aged fifteen and over." It has 154 problems arranged in ten topical chapters (e.g., Fibonacci sequences), which come from number theory, geometry, combinatorics, and elementary group theory. Brief sample: "Solve the functional equation $xf(x) + 2f(-1/x) = 3$ for real numbers $x \neq 0$."

NEWS AND LETTERS

Guidelines for Authors

General information *Mathematics Magazine* is an expository journal of undergraduate mathematics.

Both adjectives in the preceding sentence are important. Articles submitted to the MAGAZINE should be written in a clear, lively, and inviting *expository* style. The MAGAZINE is not a research journal; papers written in the "theorem-proof-corollary-remark" style are, as a rule, unsuitable for publication. The best contributions contain examples, applications, historical background, and illustrations. We especially welcome papers that include a historical element, and ones that draw connections among various branches of the mathematical sciences, and between mathematics and other disciplines.

Every article should contain interesting mathematics, readably presented. Thus, for instance, articles on mathematical pedagogy alone, or articles that consist mainly of computer programs, are unsuitable unless accompanied by interesting mathematics.

The MAGAZINE is an *undergraduate* journal in the broad sense that it addresses both teachers and students of collegiate mathematics. Among the intended uses of the MAGAZINE is to supplement and enliven undergraduate mathematics courses, especially at the upper undergraduate level. Articles, therefore, should be attractive and accessible to nonspecialists, including well-prepared undergraduates.

Writing and Revising *Mathematics Magazine* is responsible first to its readers (most of whom are mathematical generalists), and then to its authors. A manuscript's publishability depends, therefore, as much on its quality of exposition as on its mathematical significance. Our general advice is simple: Say something new in an appealing way, or say something old in a refreshing way. But say it clearly and straightforwardly, assuming a minimum of technical background.

Good exposition in our sense is vigorous and informal, written in the active voice, and rich with helpful examples. Minimize computation; stress motivation, insight, and illustration. Illustrate your ideas with visually appealing graphics, including figures, diagrams, tables, drawings, and photographs.

First impressions are especially important. Choose a short, descriptive, and attractive title; be sure that the opening sentences provide an inviting introduction to the entire paper, explaining its importance, its purpose, and its appeal.

Our referees are asked to check for mathematical accuracy, but also to give detailed suggestions on stylistic matters. In practice, most papers require a careful revision by the author, followed by a thorough editing in our office.

References should be provided generously, since we aim to invite readers—including students—to pursue ideas further. Bibliographies may contain suggested reading along with sources actually used or cited.

Many useful general references on mathematical style and exposition are available. Several are listed at the end of these notes.

Style and format Most papers are published either as Articles or as Notes. Articles have a broader scope than Notes, and usually run longer than 2000 words. Notes are typically shorter and more narrowly focused. Articles should be divided into a few subsections, each with a carefully chosen subtitle. Notes, being shorter, are usually less formally sectioned—the normal connectives of English prose should suffice to maintain the flow of ideas. The MAGAZINE itself is a ready source of examples for both forms.

In addition to expository pieces, we accept a limited number of Math Bites, poems, cartoons, Proofs without Words, and other miscellanea. See any issue of the MAGAZINE for examples of these genres.

Manuscripts should be clearly typewritten or laser-printed, with wide margins and line-spacing. The title, author, and author's address should appear at the top of the first page. Pages should be numbered.

List references either alphabetically or in the order cited in the text; in either case, consistency is essential. Please adhere very closely to the MAGAZINE's style for capitalization, use of italics, etc. See any issue (and the references below) for examples. In particular, journal titles should be abbreviated as in *Mathematical Reviews*.

Figures and illustrations Figures and illustrations may be interspersed with text, but separate copies of all illustrations should also be supplied, both with and without added lettering. Each illustration should be numbered, and referred to by number in the text. Authors themselves are responsible for providing figures of publishable quality.

Submitting manuscripts Please submit three copies; keep a fourth copy as protection against possible loss. Manuscripts and other correspondence should be mailed to

> Paul Zorn, Editor, *Mathematics Magazine*, St. Olaf College, 1520 St. Olaf Avenue, Northfield, Minnesota 55057-1098.

Please include an E-mail address, if available.

Electronic submission Electronic submission of manuscripts and figures is possible in limited circumstances. For details, contact `mathmag@stolaf.edu`.

REFERENCES

1. American Mathematical Society, *A Manual for Authors of Mathematical Papers*, 8th edition, American Mathematical Society, 1984.

2. R.P. Boas, Can we make mathematics intelligible? *Amer. Math. Monthly* 88 (1981), 727–731.

3. Harley Flanders, Manual for *Monthly* authors, *Amer. Math. Monthly* 78 (1971), 1–10.

4. Leonard Gillman, *Writing Mathematics Well*, MAA, 1987.

5. Andrew Hwang, Writing in the age of LaTeX, AMS Notices 42 (1995), 878–882.

6. D.E. Knuth, T. Larrabee, and P.M. Roberts, *Mathematical Writing*, MAA Notes #14, 1989.

7. N. David Mermin, *Boojums All the Way Through*, Cambridge Univ. Pr., Cambridge, UK, 1990.

New Mathematical Diversions

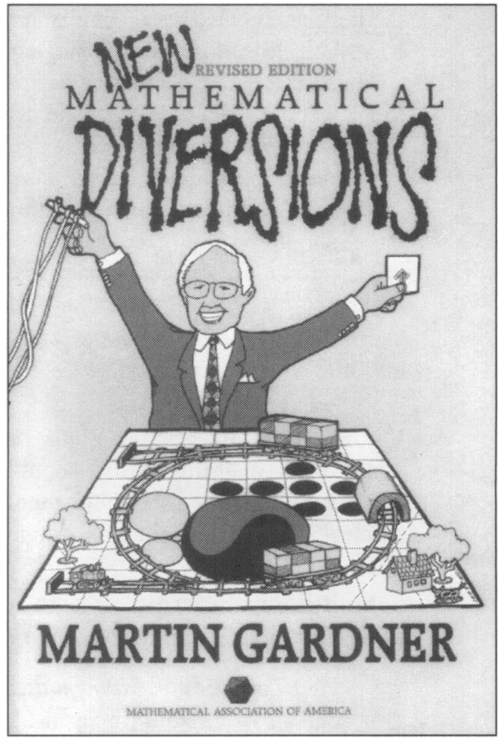
Martin Gardner

Revised Edition

Required reading for students, teachers, mathematicians at all levels, as well as interested laypersons.

This book presents twenty wonderful reprints from Martin Gardner's monthly column in *Scientific American*. Gardner tells us that his book is a book of "mathematical jokes," if "joke" is taken in a sense broad enough to include any kind of mathematics that is mixed with a strong element of fun. Readers of this book will be treated to a heavy dose of fun, and they will learn a lot about mathematics along the way.

Martin Gardner instructs us about mathematics as he entertains us with his wit and sense of the absurd. He stimulates, challenges and delights his readers. Martin Hollis (in *New Scientist*) says it best when he says of Gardner's work, "Should you ever need to explain subatomic particles to a Stone-age man, send for Martin Gardner...He leaves open questions open, conveys the thrill of the chase and deals flawlessly with hard and simple ideas alike."



Some of the topics you will find in *New Mathematical Diversions* are:

- Group Theory and Braids
- The Games and Puzzles of Lewis Carroll
- The Transcendental Number Pi
- H.S.M. Coxeter
- Polyominoes and Fault-Free Rectangles
- Euler's Spoilers: The Discovery of an Order-10 Graeco-Latin Square
- The 24 Color Squares and the 30 Color Cubes
- Bridg-it and Other Games

Answers are provided for these problems, as well as references for further reading and a bibliography. The Postscript section provides updates to the problems.

272 pp., Paperbound, 1995
ISBN 0-88385-517-8
List: $19.95
MAA Member: $16.50
Catalog Code: DIVER/JR

ORDER FROM:
THE MATHEMATICAL ASSOCIATION OF AMERICA
1529 Eighteenth Street, NW Washington, DC 20036
1-800-331-1622 (301) 617-7800 (301) 206-9789

Membership Code: ____ ____ ____ ____ ____ ____

Name ______________________

Address ______________________

City ______________________

State ____________ Zip ____________

QTY.	CATALOG CODE	PRICE	AMOUNT
______	DIVER/JR	______	______
______	______	______	______
		TOTAL	______

Payment ☐ Check ☐ VISA ☐ MasterCard

Credit Card No. ______________________ Expires ___/___

Signature ______________________

CONTENTS

ARTICLES

NOTES

PROBLEMS

REVIEWS

NEWS AND LETTERS

THE MATHEMATICAL ASSOCIATION OF AMERICA
1529 Eighteenth Street, NW
Washington, D.C. 20036